Resource allocation

Problem presented by

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Dstl

Problem statement

A military commander has to allocate resources to different courses of action in such a way as to achieve some desired set of outcomes. His view of what can be achieved by the different actions will vary with time as he sees the outcomes of his earlier choices. The outcomes also have intrinsic uncertainty. How can a mathematical model be made of this? At a different level, a defence organisation faces a corresponding problem when it has to partition funding between the armed services and defence technology research and development. The Study Group focused on the latter question, showed how it can be formulated as a problem in optimal control with delay, and studied examples of its behaviour numerically.

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1 Executive Summary

This report discusses the problem of the allocation of resources: how should an organisation (such as MOD) invest bearing in mind the long term delay for the realization of investment strategies, and how might this apply in times of increasing budgetary constraints? After making certain simplifying assumptions, the Study Group constructed a prototype model based on the method of Optimal Control. This allows the decision maker to investigate the impact of particular investment strategies over a period of years, the impact being measured in terms of "quality" or "capability". Interventions can be designed so that "quality" (Q) is maximized at a particular time, or so that the average quality over a given time interval is maximized. Both of these approaches are explored. This model shows reasonable behaviour when tested over a parameter set. It could be used as part of a systems approach to the defence budget as a whole, but the method itself is scalable to smaller (or larger) resourcing conundrums.

2 Introduction

In this report we detail several problems of "resource allocation". The first concerns high-level financial planning for the Ministry of Defence (MOD). Each year, the budget for the Ministry is set by the Treasury and depends on current policy[1]. The Minister then decides on the proportion of the budget that is spent on research and development into new equipment and methods (which we will call "future equipment and research"), and how much is given to each of the three services to pay for personnel, current equipment, works and stores. The proportion given to each of the services, and to future equipment and research is determined by the perceived quality of each of the services. Of course, there is also a pressure on the Treasury from other government departments, which can also impact on the allocation. A simplified schematic of the situation is shown in figure 1.

In this figure, we denote by £B the allocation of funds for the year, normalised by inflation, given to the minister by the Treasury. The proportion of this allocation given to each of the services and to future equipment and research are denoted by the λ_is. Spending on future equipment and research incurs a delay between the spend and the services receiving any benefit, and thus feeds into the future quality of the three services, rather than affecting them instantaneously.

The concept of quality of each of the armed forces is difficult to quantify. It might be thought of as "value for money" or "effectiveness", in this context.

In this problem, the background financial situation is one of, at best, modest growth, and is more likely to be a reduction in real terms, known as "managed decline". Dstl would like to have a model that describes how the decision maker allocates a limited set of resources over time, subject to uncertainty about the level of resource in future years.

[1] Money for specific short-term military operations comes from a government contingency fund and so doesn't contribute to the standard allocation.

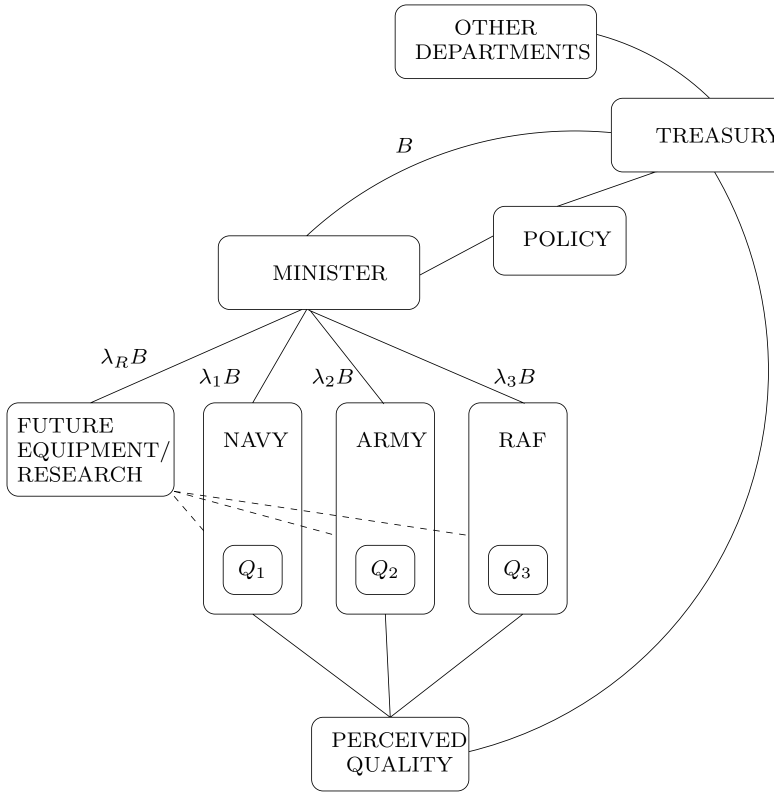


Figure 1: Simplified schematic showing the big picture for resource allocation for the MOD.

In particular, we would like to examine whether there is a way to alter the resources allocated to keep the overall "quality" as high as possible, given a developing financial situation.

The second problem of interest is that of allocation of air assets to tasks. In this problem, a commander must decide how much effort to put into different tasks during an operation, for example, reconnaissance, combat air patrols, media campaigns etc, given limited information about what other players are doing, and perceived knowledge about the importance of various actions.

3 Making the best of managed decline

We reduce the situation described above to the simplest one possible that still has the same features, in which we lump the three services together into the "Armed Forces", and we look at the competition between "service" and "research". In this case, the normalised budget £B, is split so that £$(1-\lambda(t))B$ and £$\lambda(t)B$ are the proportions of the budget going to research and the service, respectively. A schematic of the reduced problem is shown in figure 2. We take the simplistic, yet reasonable, view that the

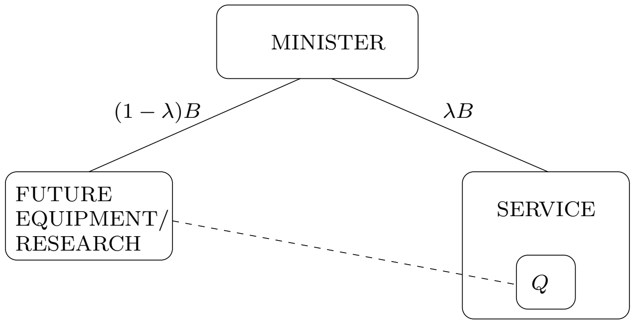


Figure 2: Schematic showing the simplified picture for resource allocation for the MOD

research quality Q_R depends directly on its current level of funding, so that

$$Q_R = (1-\lambda)B, \tag{1}$$

although a more reasonable model might be to assume that the evolution is proportional to the previous investment as well as the current quality.

We suppose that the quality of the service increases due to a combination of all the previous investment in research (which depends on how long it takes for the research to cause benefit to the service) and recruitment and retention of good quality staff. Note that the perceived required quality of the service for a given budget affects recruitment; if the quality is at its required level, no more recruiting is needed (all other things being equal). Attrition (due to wars, redundancy etc) will decrease the quality. Thus we write that

$$\frac{dQ}{dt} = \left[c\int_{-\infty}^{t} \gamma(t-s)(1-\lambda(s))B(s)ds\right]Q + \beta Q\left(1-\frac{Q}{\chi\lambda B}\right) - dQ, \tag{2}$$

where c is the effectiveness rate for the previous investment, $\gamma(t-s)$ is a measure of the impact at time t of the research quality $Q_R(s)$ that occurred in the past at time $s<t$. (This is normalised so that $\int_0^\infty \gamma(\tau)d\tau = 1$.) Also β is the recruitment rate, χ relates the spend to the perceived required quality for that spend, and d is the attrition rate. In the first term on the right, we have written the integral extending to $-\infty$ in order to represent the cumulative influence of all work in the past: in practice one might have $\gamma(\tau) = 0$ for $\tau > \tau_0$ and then the integral in (2) would just be from $t-\tau_0$ to t. Also it should be noted that the first term on the right of (2) involves the product of B and Q. This is intended to capture the fact that the quality of the service is governed by a product of the manpower involved in the service and the technology available to each man. A more sophisticated model might attempt to disentangle these rather better, with research funding affecting the quality via a measure of the technology per man, but for the moment we proceed with the model as stated in (2). We note that Middleton et al. [2] make the same point in the words:

> if R&D shows you how to make a better tank, then you derive more financial return from that R&D if you make 1000 tanks than if you only make 10.

Note that we can combine the last two terms in the RHS of (2) to give

$$\beta_0 Q\left(1-\frac{Q}{\chi_0\lambda B}\right), \tag{3}$$

where $\beta_0 = \beta - d$ is the excess of recruitment over attrition and χ_0 is such that $\chi_0 B$ is the quality arising out of pouring all the money into the service.

Finally, we have to pick an appropriate form for the budget. We set

$$B = e^{\mu t}, \tag{4}$$

where μ is a small constant (because the Treasury doesn't like to alter budgets by much) that can be of either sign.

Assuming that the budget B and the allocation λ are constant, the steady state of (2) is given by

$$Q_{ss} = ((\beta - d) + c(1-\lambda)B)\,\frac{\chi\lambda B}{\beta}. \tag{5}$$

We can easily see that direct funding to the service causes a linear increase in quality, but the research funding has a quadratic effect. The maximum value of Q_{ss} is achieved when

$$\lambda = \frac{1}{2} + \frac{\beta - d}{2Bc}, \qquad \text{if} \qquad Bc \geq \beta - d \geq -Bc, \tag{6}$$

$\lambda = 1$ if $\beta - d \geq Bc$ and $\lambda = 0$ if $\beta - d \leq -Bc$. Thus if the budget is small, research ineffective or recruitment buoyant, the best strategy is to put all the money into the service and none into research. Above the critical threshold, the best strategy is to split the budget between the two, with the best strategy being to spend half on research and half on the service if the budget becomes huge. Finally, if the attrition rate exceeds the sum of the recruitment rate and the research effectiveness rate ($d > \beta + Bc$), the steady state is such that there is no service left *however* the budget is allocated.

The solution to (2) gives us the quality of the armed service at a particular time. We have then, to determine our measure of "overall quality". There are two obvious choices (i) what matters is the end point of some spending rounds, in which case we would be interested in maximising $Q(T)$; (ii) what matters is the total quality across the spending rounds, in which case we would be interested in maximising $\int_0^T Q$.

There are then two ways to proceed. One is to pick a constant, or piecewise constant, λ, find $Q(T)$ or $\int_0^T Q dt$ and then iterate to find the λ that gives the biggest Q. The second approach is to use the calculus of variations to formulate the appropriate Euler-Lagrange equations and then determine the $\lambda(t)$ that maximises $\int Q$. In the remainder of section 3 we will adopt the former approach, while in section 4 we will adopt the latter.

3.1 A simple form for γ

We can make progress with the problem when we assume that the previous research has an effect only at a time τ later. We set $\gamma(z) = \delta(z - \tau)$ and, in this case, (2) reduces to

$$\frac{dQ}{dt} = c\,(1 - \lambda(t-\tau))\,B(t-\tau)Q + \beta Q\left(1 - \frac{Q}{\chi\lambda B}\right) - dQ. \tag{7}$$

It is clear that we must couple this equation with an initial value for Q, and that in order to solve the forward problem, we must also specify the level of research investment prior to the period under investigation.

3.1.1 Maximise $Q(T)$

We look at the two simplest cases, namely (i) we spend an equal amount on the service as we do on the research (and take $\lambda = 1/2\ \forall t$) and (ii) that we spent an equal amount in the past, and now we choose to spend everything on the service in the future. We set the funding level to be the same for all time, and we show these two scenarios in figure 3. We see that, initially, the quality of the service increases more rapidly when no money is being spent on research (recruiting increases, and there's still the benefit of previous investment in research). Then when the previous research has all kicked in, the quality declines. The long term behaviour results in scenario with equal spend giving higher quality. Of course, if we are interested in a final time which is shorter than the "incubation" time for the research, the best strategy is to put all the resources into the service.

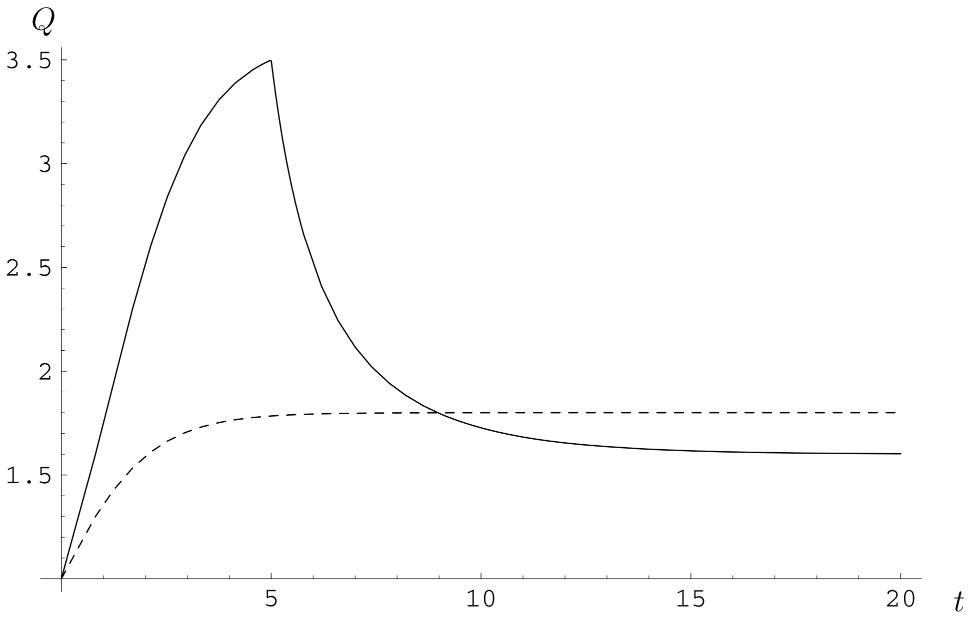


Figure 3: Graph showing two possible scenarios: equal spend on the service and research (dashed) and no further expenditure on research (solid). Parameters: $c = 1$, $\beta = 0.5$, $\chi = 2$, $d = 0.1$, $\tau = 5$, $\mu = 0$, $T = 20$, $Q(0) = 1$.

As a further example, suppose we are allowed to change the value of λ now and again after $T/2$ years, setting $\lambda = \lambda_1$ at the start ($t = 0$) and $\lambda = \lambda_2$ at the end of the period. (As before, we assume $\lambda = 1/2$ historically.) To illustrate this we show two possible options in figure 4, $\lambda_1 = 0.1$ for the first 10 years, and then $\lambda_2 = 0.9$ for the final 10 years (dashed) and vice versa (solid). We note that, with this choice of parameters, Q has almost achieved its steady state at the end of the second cycle. This has the consequence that, in fact, the choice of λ_1 barely matters; $Q(T) \approx Q_{ss}$ where Q_{ss} is the steady state solution of (7) with $\lambda = \lambda_2$.

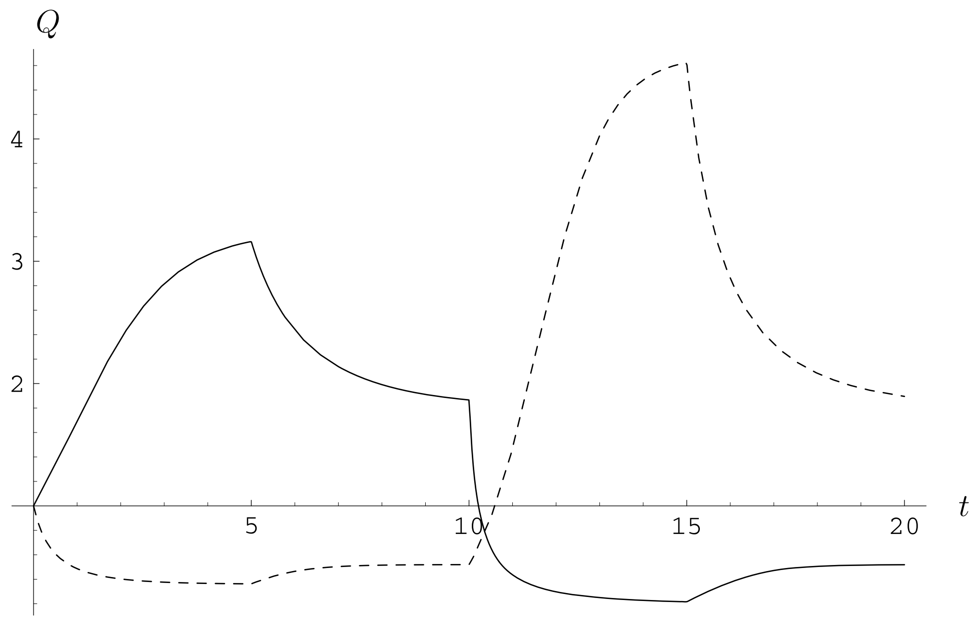


Figure 4: Graph showing two possible scenarios $\lambda_1 = 0.1$, $\lambda_2 = 0.9$ (dashed) and $\lambda_1 = 0.9$, $\lambda_2 = 0.1$ (solid). The other parameters used were $c = 1$, $\beta = 0.5$, $\chi = 2$, $d = 0.1$, $\tau = 5$.

We illustrate a case where the value of Q at $t = T$ is significantly altered by both λ_1 and λ_2 in figure 5. We find that for the parameters chosen, the final quality is maximised by picking $\lambda_1 = 0.181$ and $\lambda_2 = 0.617$.

Note that the parameters used in this section have been chosen arbitrarily to aide exposition. Determining appropriate values for the parameters would be a considerable research exercise in itself.

3.1.2 Maximise $\int_0^T Q dt$

We consider again the case where, in the past, an equal amount has been spent on the service as on research, and that we have λ_1 for the first $T/2$ years and λ_2 for the second $T/2$ years. We show the value of $\int_0^T Q dt$ in figure 6, using the same parameters as used to generate figure 5.

We find that the maximum is now achieved by picking $\lambda_1 = 0.355$ and $\lambda_2 = 1$.

Of course, these are only illustrative examples of the behaviour, and, indeed, we wish to change λ much more often. In the limit in which the time over which we wish to

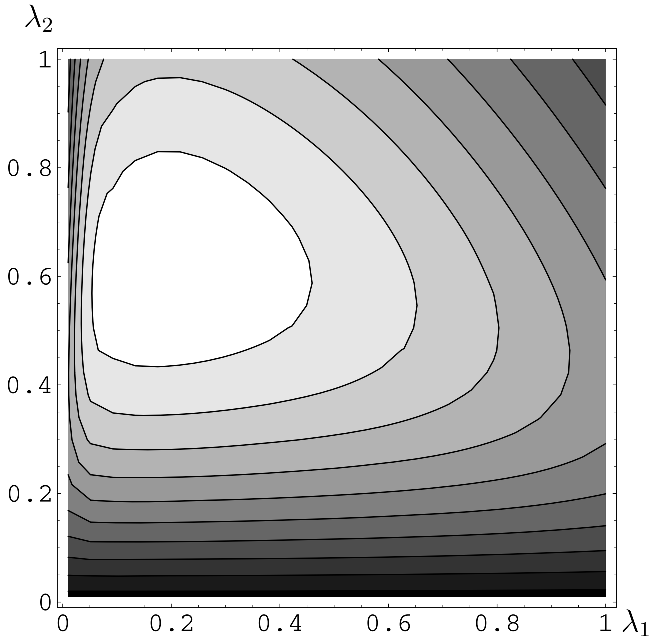


Figure 5: Graph showing how the final quality of the service varies as a function of λ_1 and λ_2. The higher the value, the lighter the colour. The other parameters used were $c = 0.5$, $\beta = 0.1$, $\chi = 2$, $d = 0.1$, $\tau = 5$.

consider the quality is much greater than the time after which we are allowed to alter the allocation, we can allow λ to vary continuously with time, and find the optimal λ from the calculus of variations. We show this approach in the next section.

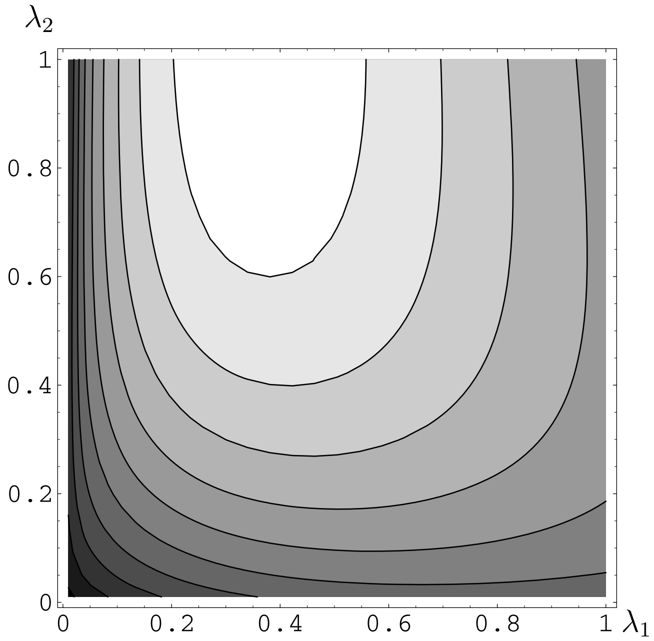


Figure 6: Graph showing how the overall quality of the service varies as a function of λ_1 and λ_2. The higher the value, the lighter the colour. The other parameters used were $c = 0.5$, $\beta = 0.1$, $\chi = 2$, $d = 0.1$, $\tau = 5$.

4 Optimal control and managed decline

4.1 Resource allocation and optimal control

We continue to examine the problem

$$\max_{0 \le \lambda \le 1} \int_0^T Q(t)dt \tag{8}$$

subject to the constraints

$$Q_R = (1 - \lambda)B, \tag{9}$$

$$\frac{dQ}{dt} = \left[c \int_{-\infty}^{t} \gamma(t-s) Q_R(s) ds\right] Q + \beta Q \left(1 - \frac{Q}{Q_w}\right) - dQ, \tag{10}$$

$$0 \le \lambda \le 1, \tag{11}$$

where we use Q_w as shorthand for $\chi \lambda B$. Our final piece of information is that Q_R is known for the past. This is an optimal control problem and the only nonstandard feature of the problem is that the dynamics of Q is governed by a differential-integral equation.

4.2 Optimal control under a special kernel function λ

In this section, we assume a simple expression

$$\gamma(t) = te^{-t}, \tag{12}$$

which suggests that there is a delay ($t = 2$ years "on average") when the quality of R&D makes an impact.

The paper by Middleton et al [2] does time-dependent correlation between spending on defence R&D of 10 nations since 1951 and military equipment quality during 1971–2005, and finds an average effective delay of 10–25 years.

4.2.1 Alternative formulation for the dynamics

For a given budget constraint $B(t)$ and a fixed allocation λ, the quality of the research and of service follow a system of two equations given in (9) and (10). Without the loss of generality, we can set the initial value of Q to be one. For the simple kernel function given in (12), we can introduce two new variables

$$Q_3 = \int_{-\infty}^{t} \gamma(t-\tau)Q_1(\tau)d\tau \text{ and } Q_4 = \int_{-\infty}^{t} Q_1(\tau)d\tau.$$

And it is straight forward to verify that (9) is equivalent to the following system of three ODEs

$$\frac{dQ}{dt} = -dQ + \beta Q\left(1 - \frac{Q}{Q_w(\lambda B)}\right) + cQQ_3, \tag{13a}$$

$$\frac{dQ_3}{dt} = Q_4 - Q_3, \tag{13b}$$

$$\frac{dQ_4}{dt} = Q_R - Q_4 \tag{13c}$$

with initial conditions

$$Q(0) = 1,\ Q_3(0) = \int_{-\infty}^{0} Q_R(t)\gamma(-t)dt,\ Q_4(0) = \int_{-\infty}^{0} Q_R(t)e^t dt. \tag{14}$$

We can rewrite the model into a more concise form as

$$\dot{\mathbf{Q}} = \mathbf{F}(t, u, \mathbf{Q}) \tag{15}$$

where $u = \lambda$ and

$$\mathbf{Q} = \begin{pmatrix} Q \\ Q_3 \\ Q_4 \end{pmatrix}, \ \mathbf{F} = \begin{pmatrix} -dQ + \beta Q\left(1 - \frac{Q}{Q_w(uB)}\right) + cQQ_3 \\ Q_4 - Q_3 \\ Q_R - Q_4 \end{pmatrix}.$$

4.2.2 Optimal control

The problem we need to solve is

$$\max_{0 \le u \le 1} \int_0^T Q(t)dt \tag{16}$$

subject to (15), which is a standard optimal control problem. It can be solved by introducing the Hamiltonian $\mathcal{H} = Q + \rho \cdot \mathbf{F}$ and the optimal solution can be obtained by solving the following system [1]

$$\max_{0\le u\le 1} \mathcal{H}, \quad \dot{\rho} = -\frac{\partial \mathcal{H}}{\partial \mathbf{Q}}, \quad \dot{\mathbf{Q}} = \frac{\partial \mathcal{H}}{\partial \rho} \tag{17}$$

with initial and terminal conditions

$$\mathbf{Q}(0) = \mathbf{Q}_0, \quad \rho(T) = 0. \tag{18}$$

The last set of equations required to close the problem is simply the set given earlier 15). The second set of equations is for the Lagrangian multiplier $\rho = [\rho_2, \rho_3, \rho_4]^T$,

$$\dot{\rho}_2 = -1 - \rho_2 \left[-d + \beta \left(1 - \frac{2Q}{Q_w(uB)} \right) + cQ_3 \right], \tag{19a}$$

$$\dot{\rho}_3 = -cQ\rho_2 + \rho_3, \tag{19b}$$

$$\dot{\rho}_4 = -\rho_3 + \rho_4. \tag{19c}$$

Finally, to find the maximal of the Hamiltonian, we apply the following conditions

$$0 = \frac{\partial \mathcal{H}}{\partial u} = -\rho_4 Q_1' B + \frac{\beta \rho_2 Q^2 Q_w' B}{Q_w^2}, \tag{20a}$$

$$\text{or } u = 0, \text{or } u = 1. \tag{20b}$$

4.3 Generalization

Suppose that we wish to optimise more than just the quality of the armed forces. If we were also interested in optimising the quality of the supporting military research itself then the problem statement in section 4.1 would be too simplistic. Our framework discussed in previous sections can be generalized to the optimal control in the following form

$$\max_{0\le u\le 1} \int_0^T J(t, u, Q_R, Q) dt \tag{21}$$

subject to a more general form of dynamic evolution

$$\dot{Q}_R = G_1(t, u_1, u_2, Q_R, Q), \tag{22a}$$

$$\dot{Q} = G_2 \left(t, u_1, u_2, Q_R, Q, \int_{-\infty}^{t} \gamma(t-\tau) Q_R(\tau) d\tau \right), \tag{22b}$$

where u_1 is the allocation to research and u_2 is the allocation to the service. If a special kernel (12) is used for γ, we can again convert the model with delay into a standard form

$$\dot{\mathbf{Q}} = \mathbf{G}(t, u, \mathbf{Q}) \tag{23}$$

where $u = u_2$ and

$$\mathbf{Q} = \begin{pmatrix} Q_R \\ Q \\ Q_3 \\ Q_4 \end{pmatrix}, \ \mathbf{G} = \begin{pmatrix} G_1(t, u, Q_R, Q) \\ G_2(t, u, Q_R, Q, Q_3) \\ Q_4 - Q_3 \\ Q_R - Q_4 \end{pmatrix}.$$

Note that this is possible for any γ whose Laplace transform is a rational function, *i.e.* $\gamma(t)$ is a linear combination of powers of t times exponentials.

Let Hamiltonian $\mathcal{H} = J + \rho \cdot \mathbf{F}$ and the optimal solution can be obtained by solving the following system

$$\max_{0 \le u \le 1} \mathcal{H}, \quad \dot{\rho} = -\frac{\partial \mathcal{H}}{\partial \mathbf{Q}}, \quad \dot{\mathbf{Q}} = \frac{\partial \mathcal{H}}{\partial \rho} \tag{24}$$

with initial and terminal conditions

$$\mathbf{Q}(0) = \mathbf{Q}_0, \quad \rho(T) = 0. \tag{25}$$

Here the Lagrangian multiplier has four components $\rho = [\rho_1, \rho_2, \rho_3, \rho_4]^T$.

4.4 Results and discussion

We now present numerical results for the simple case with $Q_w(B_2) = 0.5B_2 = 0.5uB$ as the capacity for the service quality. The values for other parameters are $d = 0.1,\ \beta = 1,\ c = 1$. Using $Q_R = (1-u)B$, from (20a) and (20b) we obtain

$$u = \min\left\{1, \sqrt{\frac{2\beta\rho_2 Q^2}{\rho_4 B^2}}\right\}. \tag{26}$$

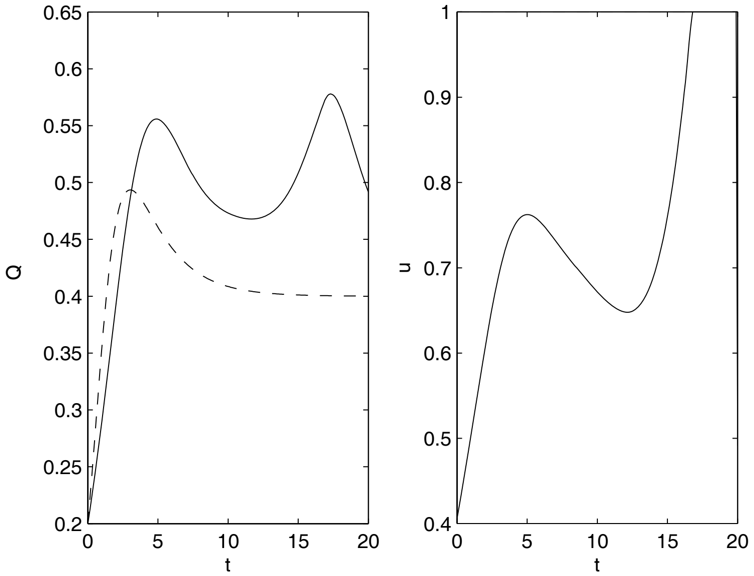
To find the optimal solution we solve the coupled systems (13a)-(13c) and (19a)-(19c) with the initial and terminal conditions (18), iteratively. We start by providing an initial guess for μ, solve (13a)-(13c) with initial condition (18), using control (26). We then solve (19a)-(19c), using the terminal condition (18). We repeat the steps above until a pre-set convergence criterion is satisfied. The initial conditions used in the computations are $Q(0) = 0.2$, $Q_3(0) = Q_4(0) = 0.5$. The total time period is $T = 20$ years.

4.4.1 Time-independent resource level

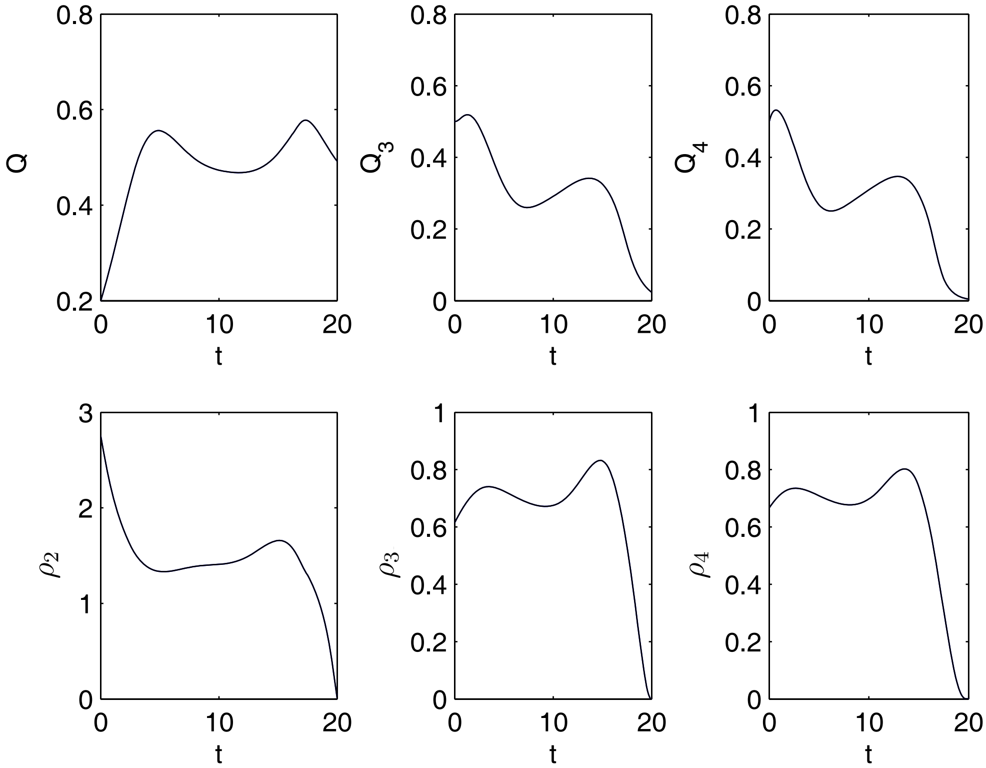
In Figure 7, we have plotted the time history of Q, u, and other auxiliary values when the available resource remains as a constant $B = 1$. For comparison purposes we have also plotted the solution of a fixed allocation ($u = 1$). The total value of the service quality $\int_0^T Q dt$ is 9.73 and 8.28 for the optimal and fixed allocation strategies, respectively. Therefore, it is beneficial to go through the optimization exercise.

4.4.2 Declining resource level

In Figure 8, we have plotted the time history of Q, u, and other auxiliary values when the available resource is assumed to decline with small μ such that we can linearise $B = e^{\mu t}$ to get $B = 1 - 0.5t/T$. Again, for comparison purposes we have also plotted the solution of a fixed allocation ($u = 0.5$). The total value of the service quality $\int_0^T Q dt$ is 7.20 and 6.02 for the optimal and fixed allocation strategies, respectively. Once again, the overall service quality is improved by investing in R&D strategically.

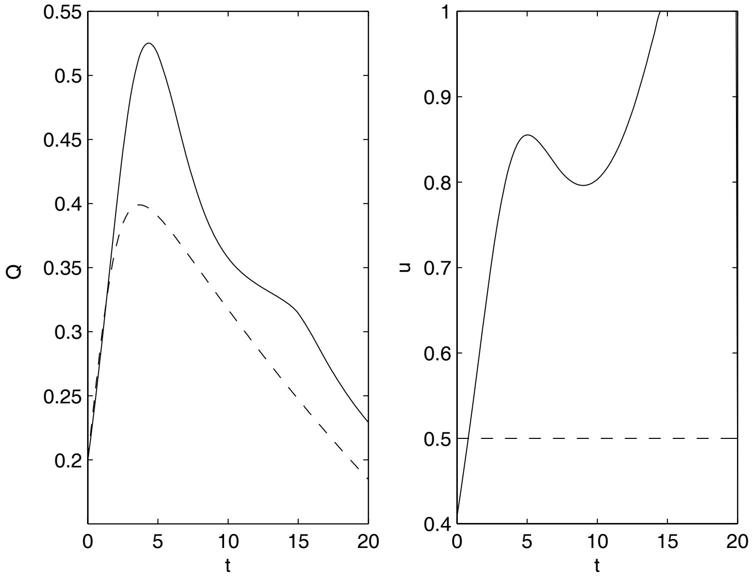


(a)

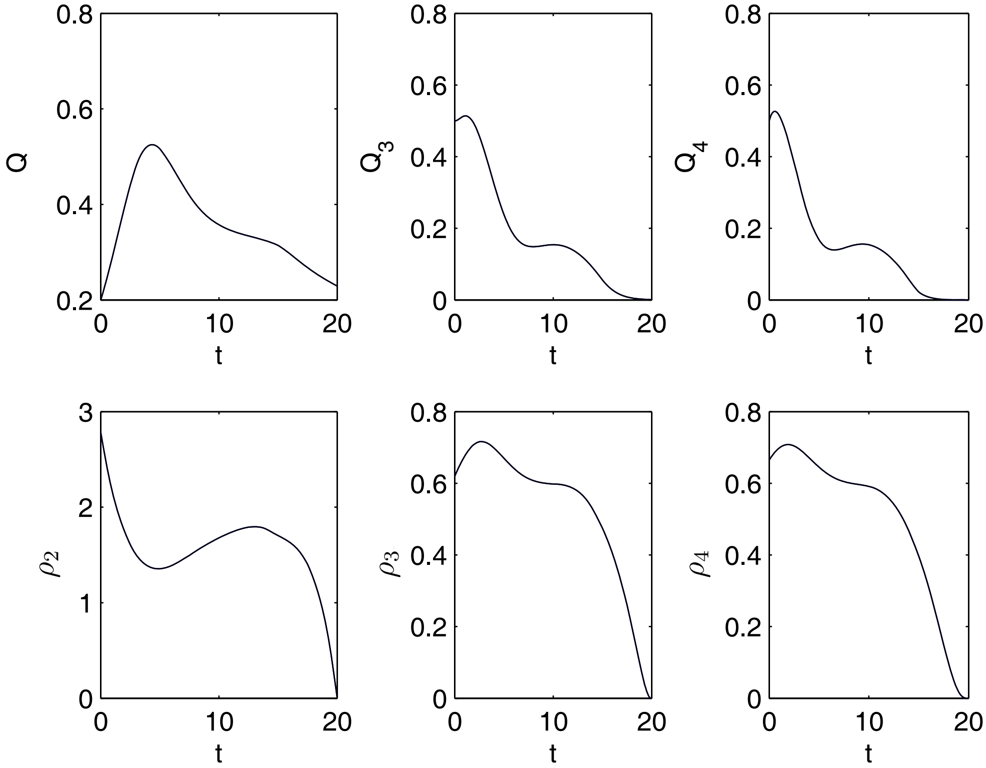


(b)

Figure 7: Solution for time-independent budget (resource). (a) Q and u where the solid line is the optimal solution and the dashed line is for a fixed allocation strategy $u = 1$; (b) Q and auxiliary variables for the optimal allocation case.



(a)



(b)

Figure 8: Solution for a declining budget (resource). (a) Q and u where the solid line is the optimal solution and the dashed line is for a fixed allocation strategy $u = 0.5$; (b) Q and auxiliary variables.

5 Allocation of Air Assets

We also considered several problems about the allocation of air assets. In each of these cases, we merely state the problem and give a mathematical formulation, but we have not attempted to solve these problems.

5.1 Mission strategy

Firstly, we look at the problem of allocating resources to two different missions: a search and rescue mission and a troop deployment mission. The former mission requires intelligence gathering from a high level aircraft and then extraction using a helicopter. The latter also requires the intelligence gathering and then deployment using a helicopter. The success of each mission depends on whether the enemy intervenes.

We supposed that the surveillance aircraft detects enemy forces with a certain probability, μ_1 say, and is shot down/forced away by enemy fighters with probability s_p per minute of flight. Intelligence becomes inaccurate at the rate χ per minute.

The helicopter has a probability of being shot down of $s_{h,1}$, unless they've been told that there are enemy forces in the area, in which case they fly more stealthily and have a probability $s_{h,2} < s_{h,1}$ of being shot down.

We also need to have a model for how the enemy forces react to the presence of our forces. We suppose that the enemy are in a given area with probability r and if our forces get detected then this probability goes up with time as $r + (1-r)t/(\phi + t)$ where ϕ is a measure of the time it takes for enemy forces to mobilise.

Given positions $\mathbf{x}_{r,d}$ of the rescue and drop off points respectively, the velocities $\mathbf{v}_{a,h}$ of the aircraft and helicopter respectively, and the initial positions $\mathbf{y}_{a,h}$ of the aircraft and helicopter respectively, our question is to determine what the best strategy is for ensuring that both missions are a success.

5.2 Assignment problem

The second problem concerns the assignment of multiple assets to missions. We suppose that missions M_i need h_i helicopters and p_i planes to succeed. (Failure is deemed certain if insufficient aircraft arrive at the scene of the mission.) The air commander has a good feel for how long each mission will last. For helicopters and planes respectively, this time is $t_i^{h,p}$, drawn from normal distributions with (commander dependent) mean $T_i^{h,p}$ and standard deviation $\Sigma_i^{h,p}$. Given a total number of helicopters $\mathcal{H}$ and planes $\mathcal{P}$, and missions arriving randomly but with priorities (on some scale), the question is how to minimise the risk and get all the missions completed successfully, if we allocate $H_i \geq h_i$ helicopters and $P_i \geq p_i$ planes. With probabilities of being shot down s_h and s_p per minute, the failure probability (and hence the risk) for mission i is

$$R = 1 - \left[\sum_{k=0}^{H_i - h_i} \frac{H_i! \, (1 - s_h)^{t(H_i - k)} \, (1 - (1 - s_h)^t)^k}{k!(H_i - k)!} \right] \left[\sum_{k=0}^{P_i - p_i} \frac{P_i! \, (1 - s_p)^{t(P_i - k)} \, (1 - (1 - s_p)^t)^k}{k!(P_i - k)!} \right]. \tag{27}$$

As an example, suppose we say that an acceptable risk level is 0.05, that we need 10 helicopters, no planes, the mission is to last 10 minutes and the probability of being shot down is 0.01. We show a plot of the risk for various allocations of helicopters in figure 9. We can see that in this example the commander should assign 13 helicopters to get

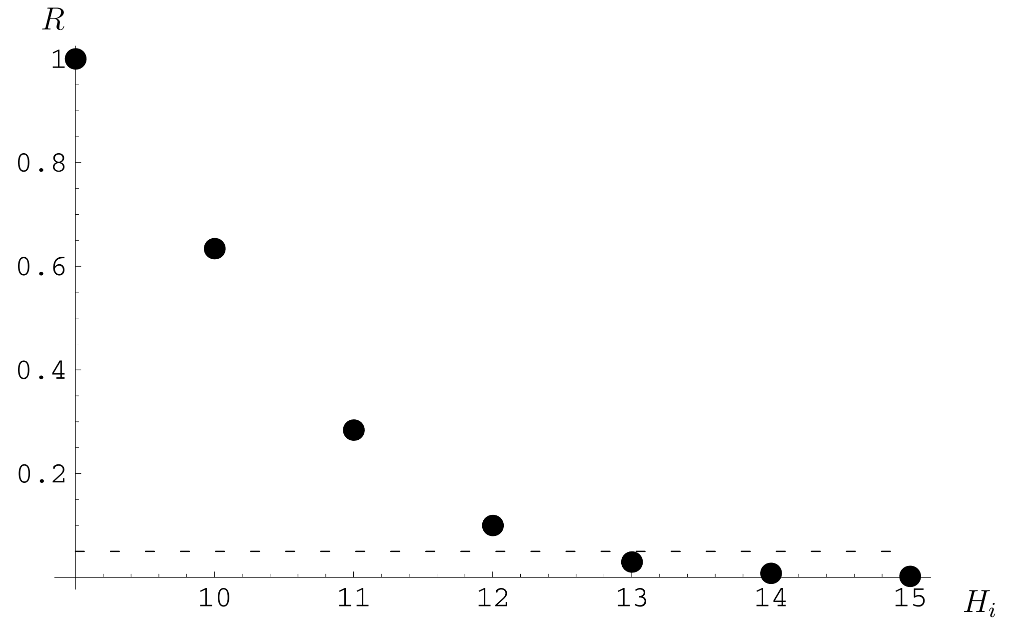


Figure 9: Graph showing the risk level against allocation of helicopters

his risk below the threshold required.

For multiple missions with the constraints on the aircraft as above, we have to consider an appropriate programming problem, using the building blocks described above.

6 Conclusions and recommendations

This report has explored the problem of resource allocation, outlining how, in this simplified case, an optimal control model can be used to assess the balance between immediate direct investment in the armed services and the longer term research investment required to maintain sufficient quality in the future. We showed two approaches: a direct calculation and iteration approach, and one using optimal control. We showed that strategic investment into research produced a service with a higher quality. Further work on the optimal control problem has been carried out in Pitcher

[3]. Of course, we have picked the simplest forms for all of the terms in our model. The proper forms could be chosen by fitting with data, using studies such as [2]. The final goal of this work should be to couple the allocation B by the Treasury into the problem by making it depend on Q. Various strategies could be considered (eg increasing spend to failing parts of the armed forces, shutting down a service if it repeatedly has low quality[2]). The allocation that will be given to the Minister in the future is uncertain, and so the models would also need to be extended to stochastic versions.

We have also laid out the groundwork for studies of allocation of air assets to various missions. Simulations would be required in order to take these models forward.

Scientifically, this problem has inspired work in an area of optimal control where the extant literature has little to say.

References

[1] Chiang, A.C., (1992) *Elements of Dynamic Optimization.* McGraw-Hill, NY.

[2] Middleton A., Bowns, S., Hartley, K. and Reid, J. (2006) *The effect of defence R&D on military equipment quality.* Defence and Peace Economics **17**, 117–139.

[3] Pitcher, A.B., (2007) *Optimal control with delay.* Transfer thesis, Oxford University.

[2]There is an obvious RAE analogy here.