Chapter 4

Density Driven Turbulent Mixing at Batch Interfaces

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4.1 Introduction

Pipelines provide an economically efficient method of transporting liquid hydrocarbon products over long distances. A pipeline may be batched, that is, it may contain a number of different types of liquids, each batch having its own fluid characteristics of density and viscosity. Liquid hydrocarbons could be fluids such as crude oil (heavy, medium, light or synthetic) or refined products (gasoline diesel fuel, aviation fuel) or volatile liquids such as Natural Gas Liquids (NGL). It is economically desirable, especially with refined products, that cross contamination between adjacent batches be minimized. If the densities of the fluids in adjacent batches are different and the flow is laminar, buoyancy forces lead to an undesirable horizontal mixing process between the two fluids with the heavier fluid forcing itself under the lighter fluid as they are both convected along the pipe by a constant pressure gradient. This type of mixing is primarily driven by density gradients and gravity, but differences in viscosity will also affect the mixing process. In the laminar flow regime, the amount of cross contamination between batches is roughly proportional to the distance that the fluids are transported. Since the average fluid velocity is constant this means that the width of the interface, W, grows proportional to time t. Pipeline companies minimize mixing by operating the pipeline in the fully turbulent flow region. Reynolds numbers of greater than 2000 are considered to be fully turbulent but most companies use a lower limit of 4000. This assures that the flow will be fully turbulent. Upon start-up and shutdown of a pipeline the flow does pass through the laminar region but this is typically a short interval. Besides, starts and stops of pipeline are generally not frequent. Experimental measurements show that this results in an interface growth rate $W \propto t^{0.56}$.

In the following analysis models are developed for the turbulent mixing and growth at a batch interface. In §4.2 the mathematical model is defined, this depends crucially on the choice of diffusion coefficient D. In §4.3 a model is analysed where D is the harmonic average of the mixing coefficients of the two pure fluids. This is likely to be a good approximation when the density difference between the fluids is small.

When the density difference is large, in the laminar flow regime fingering will occur and there will be a relatively sharp interface between the fluids. However, in the turbulent case, as gravity drives the denser fluid into the less dense one the invading fluid is immediately mixed by turbulent diffusion. This means that sharp interfaces will not exist. Instead there will be a finite mixing region where the volume fraction of each fluid

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changes from 1 to 0. In this case D will depend upon the relative concentration of the fluids. This approach leads to a degenerate diffusion problem which is analysed in §4.4.

4.2 Governing equations

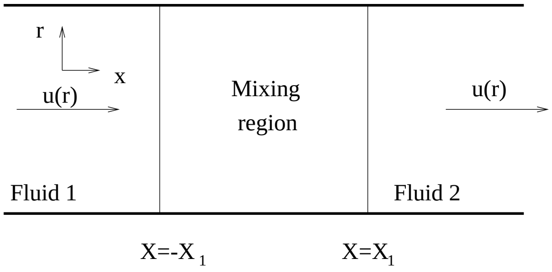

Figure 4.1: Problem configuration for $t > 0$

At an initial time $t = 0$ we consider fluid 1 occupying the section of pipe $x < 0$ and fluid 2 occupying the section of pipe $x > 0$, as depicted in see Figure 4.1. Let $c_1(x,t)$ and $c_2(x,t)$ denote concentrations (number per unit volume) of virtual particles of types 1 and 2, respectively. The concentrations of the two populations of virtual particles are assumed to be proxies or indicators for the mixing of the two fluids. The two types of particles are assumed to be distinguishable from one another and respond to the forces of the adjacent fluid elements just as if they were themselves characteristic volume elements of the fluid in its turbulent state. Assume that the fluids are incompressible so that

$$c_1(x,t) + c_2(x,t) = \bar{c}, \qquad t > 0 \ . \tag{4.1}$$

The turbulent diffusion equations for the two sets of particles are:

$$\frac{\partial c_i}{\partial t} + \mathbf{u} \cdot \nabla c_i = \nabla \cdot (D_i \nabla c_i) \ , \tag{4.2}$$

where D_i is the turbulent diffusion coefficient in fluid i and $\mathbf{u} = (u, w)$ is the fluid velocity. Finally, we assume initial concentrations

$$c_1(x,0) = \begin{cases} \bar{c}, & x < 0 \\ 0, & x > 0 \end{cases} \tag{4.3}$$

and

$$c_2(x,0) = \begin{cases} \bar{c}, & x > 0 \\ 0, & x < 0 \end{cases} \ . \tag{4.4}$$

If $\alpha_1(x,t)$ and $\alpha_2(x,t)$ represent the volume fractions of fluid 1 and fluid 2 respectively, because of the identification discussed above, we set

$$\alpha_1 = \frac{c_1}{c_1 + c_2} = \frac{c_1}{\bar{c}} \qquad \alpha_2 = \frac{c_2}{c_1 + c_2} = \frac{c_2}{\bar{c}}. \tag{4.5}$$

Since $\bar{c}$ is a constant the turbulent diffusion equations may be rewritten as

$$\frac{\partial \alpha_i}{\partial t} + \mathbf{u} \cdot \nabla \alpha_i = \nabla \cdot (D_i \nabla \alpha_i) \ . \tag{4.6}$$

The turbulent diffusion coefficient in (4.6) must be determined experimentally for the fluids in question. The turbulent velocity $\mathbf{u}$ may be estimated from available existing results. This is discussed below.

With a 20" (0.508m) pipe, an average velocity of 5mph (2.235 m/s) and the fluid dynamic viscosity range supplied by Enbridge $\nu \in [0.1, 170]$cS ($\nu \in [10^{-7}, 1.7 \times 10^{-4}]$m^2/s) the Reynolds number for the flow is in the range $R_e \in [6.68 \times 10^3, 1.1 \times 10^7]$. The flow is therefore turbulent for all cases of interest. Experimental work shows that for turbulent flow, the velocity component along the pipe is closely approximated by

$$u = U_{max}\left(1 - \frac{r}{R}\right)^{1/n} , \tag{4.7}$$

where $n = 6, 7, 10$ for Reynolds numbers $R_e \approx 4 \times 10^3, 1.1 \times 10^5, 3.2 \times 10^6$ [1]. The wide variation calculated for the Reynolds number in the present case indicates that the appropriate value for n will depend strongly upon the fluids in question.

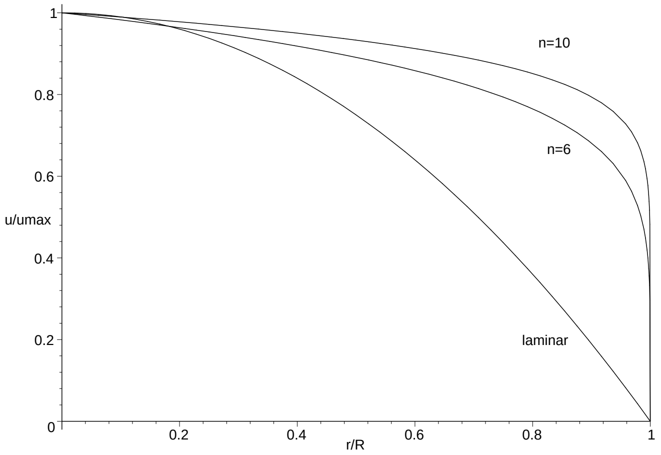

Figure 4.2: Velocity profile from $r = 0$ to R

The velocity profiles corresponding to values of $n = 6$ and 10 are shown in Figure 4.2. The laminar, parabolic profile is also plotted. When the flow is turbulent, in the vicinity of $r = R$, there is a boundary layer, away from this region the velocity varies only very slowly. For this reason u is well approximated by a constant value U_a, except for in the vicinity of $r = R$. However, since this boundary layer occupies only a small proportion of the cross-section it will not have a significant effect on the bulk properties and so can be neglected. The experience of pipeline companies indicates that for pipes of diameter greater than 12" the boundary layer is not hydraulically significant.

Since the flow is turbulent, mixing across the pipe happens over a very short time-scale. This means that any radial variations are quickly smoothed out and so radial derivatives will be negligible compared to derivatives along the pipe. Further, the (time) average velocity in the radial direction, w, must be zero. The diffusion equations may therefore now be written in a considerably simpler form

$$\frac{\partial \alpha_i}{\partial t} + U_a \frac{\partial \alpha_i}{\partial x} = \frac{\partial}{\partial x}\left(D_i \frac{\partial \alpha_i}{\partial x}\right) , \tag{4.8}$$

where radial variations have been neglected.

The heart of the current problem lies in determining the turbulent diffusion coefficients D_i. In the following section a standard diffusion model will be employed. The diffusion coefficient will be chosen as the harmonic average of the two diffusion coefficients for the pure fluids. At either end of the interface the harmonic average reduces to the value for a single fluid. As with standard diffusion problems this will imply an infinite speed of propagation and the width of the mixing zone will be infinite. This is the essence of the approach adopted in [2]. In this case the mixing zone must be defined by the region where the concentration of each component is above some minimum value. This model will be most appropriate when there is very little variation in the densities of the fluids in question. In §4.4, it will be assumed that the amount of mixing depends on the relative concentration of the components. At the centre of the mixing zone, the mixing will be greatest, due to the high concentration of each component, away from the centre the mixing will decrease and reach zero at the edges of the mixing zone.

4.3 Mixing of fluids with a similar density

In this section the diffusion coefficient will be chosen as the harmonic average of the two diffusion coefficients of the pure fluids:

$$D = \frac{1}{\frac{1}{D_1}\alpha_1 + \frac{1}{D_2}\alpha_2} \,. \tag{4.9}$$

Note that $D \to D_1$ as $\alpha_2 \to 0$ and $D \to D_2$ as $\alpha_1 \to 0$. The rationale for using the harmonic average comes from an analogy with diffusion in homogenized media. The effective diffusion coefficient in this case is well known to be the harmonic average of the diffusion coefficients over a period length of the microscopic media.

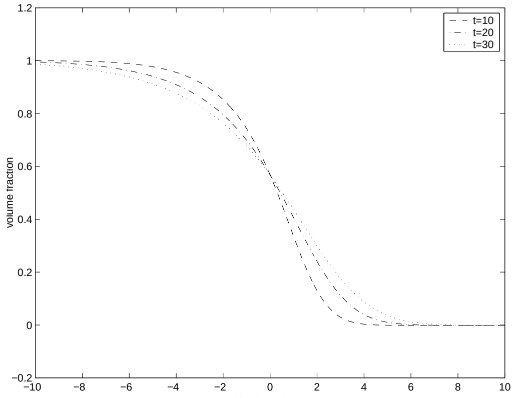

Figure 4.3: Volume fractions, α_1, when $t = 10, 20, 30$; D_1=1/2 and D_2=1/10

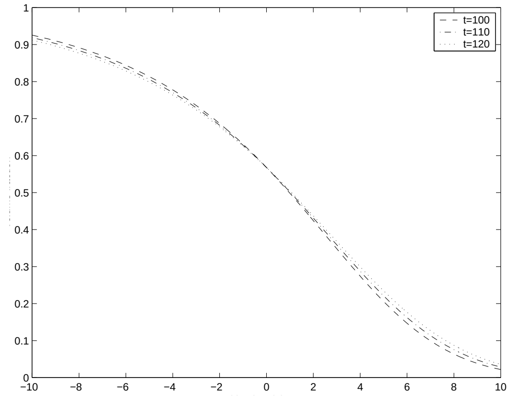

Figure 4.4: Volume fractions, α_1, $t = 100, 110, 120$; D_1=1/2 and D_2=1/10

Shifting co-ordinate system to move with the fluid $X = x - U_a t$ the problem is now defined by

$$\frac{\partial \alpha_1}{\partial t} = \frac{\partial}{\partial X}\left(D\frac{\partial \alpha_1}{\partial X}\right) \tag{4.10}$$

$$\alpha_1(X,0) = \begin{cases} 1, & X < 0 \\ 0, & X > 0 \end{cases} \tag{4.11}$$

$$\alpha_1(X,t) \to 1 \quad \text{as} \quad X \to -\infty \quad , \quad \alpha_1(X,t) \to 0 \quad \text{as} \quad X \to \infty \tag{4.12}$$

where

$$D = \frac{1}{\frac{1}{D_1}\alpha_1 + \frac{1}{D_2}(1-\alpha_1)} = \frac{D_1 D_2}{D_2\alpha_1 + D_1(1-\alpha_1)} \ . \tag{4.13}$$

4.3.1 A similarity solution

If we set $\alpha_1(x,t) = f(\eta)$ where $\eta = \frac{X}{\sqrt{t}}$ we find that f must satisfy the two point boundary value problem

$$\left\{ \frac{D_1 D_2}{D_2 f + D_1(1-f)} f' \right\}' + \frac{1}{2}\eta f' = 0, \qquad \lim_{\eta\to-\infty} f(\eta) = 1, \qquad \lim_{\eta\to\infty} f(\eta) = 0. \tag{4.14}$$

The solution of this system will be a good approximation to that of (4.10)-(4.12) for large times.

With a constant diffusion coefficient a similarity solution exists for all time [3]

$$f = \frac{1}{2}\text{erfc}\left(\frac{\eta}{\sqrt{2D}}\right) \ . \tag{4.15}$$

When the diffusion coefficient is defined by (4.13) the problem must be solved numerically. With values of $D_1 = 0.5$, $D_2 = 0.1$ solutions for times $t = 10, 20, 30$ are shown in Figure 4.3, for $t = 100, 110, 120$ solutions are shown in Figure 4.4. Two solutions are compared in Figure 4.5 to demonstrate the difference in concentrations when the diffusion coefficients are swapped. This highlights the fact that for any two fluids the concentration profile in the mixing zone depends on the order of the fluids, *i.e.* which is driving and which is being driven.

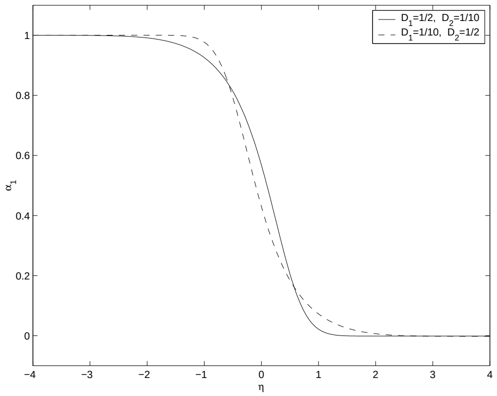

Figure 4.5: Solution to (4.14), two cases: D1=1/2 and D2=1/10 and vice-versa

If we arbitrarily define the position of the trailing edge, $x_t(t)$, and the position of the leading edge, $x_l(t)$, of the mixing zone to be locations where the composition is 90% fluid 1 and 90% fluid 2, respectively, then

$$x_t(t) = \eta_1\sqrt{t} + U_a t$$

and the leading edge

$$x_l(t) = \eta_2\sqrt{t} + U_a t$$

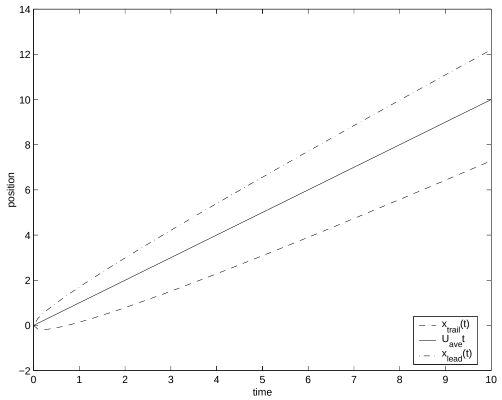

Figure 4.6: $x_t = U_a t + \eta_1 \sqrt{t}, x_l = U_a t + \eta_2 \sqrt{t}, x_c = U_a t; D_1 = 1/2, D_2 = 1/10$

where η_1 and η_2 are determined from $f(\eta_1) = 0.9$ and $f(\eta_2) = 0.1$. Graphs of $x_t(t)$ and $x_l(t)$ are illustrated in Figure 4.6 for the case $D_1 = 0.5$ and $D_2 = 0.1$ (so that the efficiency of the turbulent mixing is greater in fluid 1 than it is in fluid 2). The effect of this is shown by the greater thickness of the mixing zone below the line $x = U_a t$ than above.

In the case that the leading fluid is more efficient at mixing than the trailing fluid ($D_1 = 0.1$ and $D_2 = 0.5$) the situation is reversed with the thickness of the mixing zone above the line $x = U_a t$ greater than the thickness below the line.

4.4 Density driven mixing

The model of the previous section is based on turbulent diffusion. It has infinite speed of propagation and so cannot describe a finite interface. An alternative approach, discussed in §1, is a density driven mixing model. This requires a turbulent mixing coefficient which is zero at either end of the interface, reflecting the fact that no mixing occurs beyond this point. Close to an interface, the mixing will be small due to the small amount of one of the fluids. The mixing then increases to a maximum at the centre and decreases again as the other interface is approached. The simplest model which captures these features is $D_i = D_0 \alpha_1 (1 - \alpha_1)$; because the fluids are well-mixed the diffusion coefficient is the same in each phase. The value of D_0 will be constant for each pair of fluids, it is expected that D_0 will vary with the viscosity and density difference between the fluids, in particular it must increase as the density difference increases. It seems reasonable to assume $D_0 = D_0(R_{e_1}, R_{e_2})$. As highlighted by the results of the previous section D must be calculated for each pair of fluids and with every fluid driving and being driven. The problem is now governed by

$$\frac{\partial \alpha_1}{\partial t} = \frac{\partial}{\partial X}\left(D_0 \alpha_1 (1 - \alpha_1) \frac{\partial \alpha_1}{\partial X} \right) , \qquad (4.16)$$

where, once again $X = x - U_a t$. Another simple model for the mixing coefficient is $D = D_0 \partial \alpha_1 / \partial x$. However, this leads to an interface which grows with $t^{1/3}$, whereas experiments indicate the growth is approximately $t^{0.56}$ so this model will not be discussed further.

Equation (4.16) may be solved numerically, alternatively a lot of useful information may be gained from the similarity solutions of the following section. Note, equation (4.16) is invariant under the transformation $\alpha_1 \to 1 - \alpha_1$, so there is symmetry about the origin. The problem therefore only requires solving on one side of the line $X = 0$.

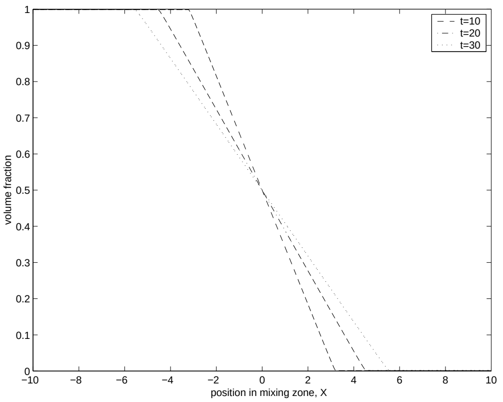

Figure 4.7: Concentration, $\alpha_1(x,t)$, when t = 10,20,30; D_0=1

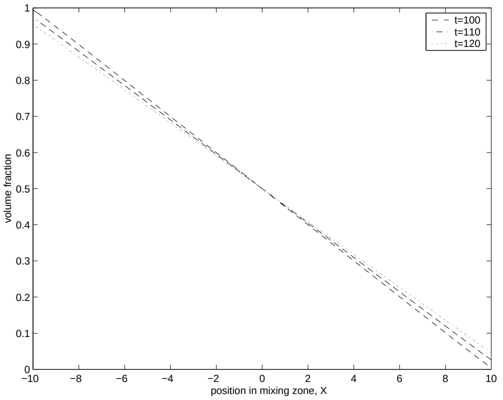

Figure 4.8: Concentration, α_1, when t = 100,110,120; D_0=1

4.4.1 Similarity solution

Employing the similarity substitution $\eta = X/\sqrt{D_0 t}$, equation (4.16) becomes

$$-\frac{1}{2}\eta\alpha_1' = \frac{\partial}{\partial\eta}\left(\alpha_1(1-\alpha_1)\alpha_1'\right) \ . \tag{4.17}$$

Appropriate boundary conditions are

$$\alpha_1(0) = \frac{1}{2} \qquad \alpha_1''(0) = 0 \qquad \alpha_1(\eta_1) = 0 \ , \tag{4.18}$$

where η_1 is the unknown position of the edge of the mixing zone. The first two conditions arise from symmetry considerations.

An exact solution to this problem is $\alpha_1 = (1-\eta)/2$. That this is the appropriate solution is confirmed by the numerical solutions of (4.17)-(4.18) shown in Figures 4.7, 4.8. Note, the ordinate in these figures is X, when η is taken as the ordinate all the curves collapse onto a single line, as shown in Figure 4.9. From this figure it is clear that the width of the mixing zone is $2\eta_1 = 2$.

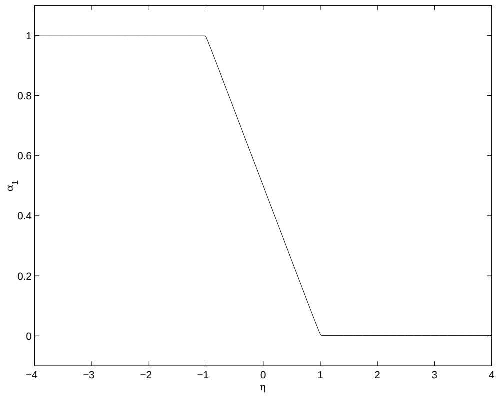

Figure 4.9: Solution to (4.17)-(4.18)

4.4.2 Width of interface

The width of the interface is simply given by $W = 2X_1$, where X_1 is the position of the leading edge, $X_1 = \eta_1\sqrt{D_0 t}$. The exact solution of the previous section shows $\eta_1 = 1$, and the interface width is therefore

$$W = 2\sqrt{D_0 t}\ . \tag{4.19}$$

This indicates that the interface grows proportional to $t^{0.5}$ and the constant of proportionality is $2\sqrt{D_0}$. The numerical results of Figures 4.7, 4.8, where D_0 is set to 1, have an interface width $2\sqrt{t}$. Experimental results given in [2] indicate $W \sim t^{0.56}$. A plot of the growth of the interface with time is shown in Figure 4.10.

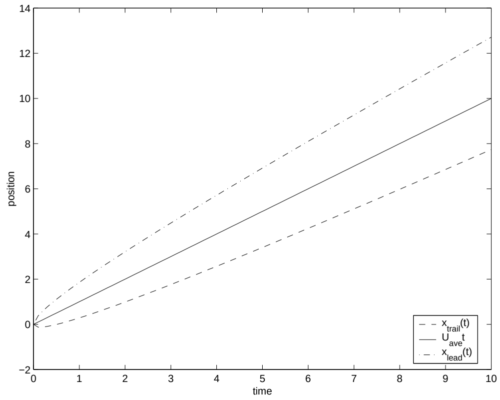

Figure 4.10: $x_t = U_a t + \eta_1\sqrt{t}, x_l = U_a t + \eta_2\sqrt{t}, x_c = U_a t; D_0 = 1/2$

4.5 Conclusions

Two models have been discussed in this paper. The first model will be most relevant to flows where the density difference between the fluids is small. In this case the diffusion/mixing coefficient is related to the diffusion coefficients in the pure fluids, D_i. For the analysis of §4.3 the coefficient is chosen as the harmonic average of D_1 and D_2. The second model will be relevant to flows where the density difference is significant. In this case gravity drives the mixing and the diffusion coefficient is not necessarily related to D_1 and D_2.

Both models studied lead to a growth of the interface with $t^{0.5}$, which is in close agreement with the experimental results given in [2]. These indicate a growth rate proportional to $t^{0.56}$. The problem description provided by Enbridge stated that the width of the mixing zone stabilizes after a sufficiently long time. The rate of change in width of the interface in both of the current models is proportional to $1/\sqrt{t}$ (this rate would also be predicted by a constant diffusion coefficient model). As indicated by Figures 3, 4, 7, and 8 this decrease in the rate of change of mixing zone width may be considered as consistent with the stabilization observed by Enbridge within the limits of experimental error

The interface width also depends crucially upon the value of the diffusion coefficient, which must be determined experimentally. In the turbulent mixing section a very simple formula was obtained for the interface width, $W = 2\sqrt{D_0 t}$. So, the only parameter controlling the width is the diffusion coefficient. The best choice of fluids to travel together will therefore be the pair with the minimum value of D_0. The results of §4.3 highlight the fact that the mixing between any two fluids depends upon which fluid is driving and which is being driven. In light of this fact it is recommended that when determining D_0 tests are carried out on each pair of fluids and with every fluid driving and being driven. It is likely that the value of D_0 is a function of the Reynolds numbers of the two fluids, hence scaled down tests may be carried out provided the Reynolds numbers are scaled appropriately.

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