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Incorporating Estimation Error into Optimal Portfolio Allocation

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1 Problem description

In this report, we consider the problem concerned with incorporating estimation error into optimal consumption and portfolio selection in continuous time. The original optimal consumption and asset allocation problem in continuous time was solved by Merton in a series of papers [4, 5] and became widely known as "Merton's Problem". Merton made the assumption that the asset price processes $\{S_i\}_{i=1}^N$ are given by Geometric Brownian Motion (GBM), where the parameter values are known. He was able to prove that the investment opportunity set can be generated by two portfolios or mutual funds of assets, which themselves obey Geometric Brownian Motion. This result is sometimes known as a two-fund separation theorem, and does not depend on the market being in equilibrium. Merton utilized the separation theorem in his development of the Inter-temporal Capital Asset Pricing Model in [6], but the mutual fund theorem is only dependent on the assumed properties of the asset price processes.

Subsequent work on this problem has sought to generalize Merton's work in numerous ways. For instance, some authors have considered more general asset price processes than GBM, e.g. Ito processes with deterministic (and even stochastic) time-dependent drift and diffusion parameters, and other general diffusion and Markov processes, or general semi-martingales. In the latter case, the additional assumption that the market is complete (or more generally, effectively complete) is required, and the method of solution uses the so-called Cox-Huang-Pliska method [9], which involves the use of the Martingale Representation Theorem. Another generalization of the Merton problem involves the inclusion of

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stochastic income for the investor, with various degrees of generality regarding the structure of the income process and transaction costs, and with investor preferences given by expected utility functions that are non-time-separable (e.g. recursive or stochastic differential utilities), or even non-expected utility preference orderings. However, most of the published work on the consumption/portfolio allocation problem in continuous time has assumed that the parameters of the asset price processes are known with perfect certainty [3, 2, 10, 8]. In reality, these parameters must be estimated, and there will always be some measure of estimation risk [8, 10].

The result of the unavoidable nature of estimation risk is that the optimized consumption/portfolio selection strategy will only truly be optimal in the unlikely event that there is no estimation error; in all other cases, it will be suboptimal. The goal of the workshop is to formulate the optimal consumption and portfolio investment problem such that, given any data sample of the asset price processes, we have a prescription that associates to that sample an optimal strategy; note that this prescription is dynamic, since the sample will enlarge over time, likely resulting in a different optimal strategy from the previous one, going forward. To eliminate unnecessary complications, it would be easiest to work within the original Merton model, except that we wish to consider the case that investors do derive benefit from end-of-period wealth (instead of the overall consumption), and investor wealth is constrained to be nonnegative.

2 Mathematical models

To address the issue related to estimation error and investment strategies, we recall the Merton framework of optimal asset allocation in continuous time. To simplify the discussion, we consider only asset allocation and ignore consumption in this report. Following Merton's approach [4, 5], we assume that

- we have an initial wealth w_0 at time t_0;
- we can choose a combination of one risky asset (S_t) and one risk-free asset (R_t), e.g. a bond;
- S_t follows the GBM
$$\frac{S_t}{S_0} = \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right] \tag{2.1}$$
with drift μ and volatility σ;
- the risk-free asset has a rate of return r, and is given by
$$\frac{R_t}{R_0} = \exp(rt). \tag{2.2}$$

Let π_t be a fraction of wealth allocated to S_t such that the utility of wealth is maximized at the end of a fixed period, i.e.,

$$J = \max_{\pi_t} \mathbb{E}\left[u(W_T)|\mathcal{F}_t\right], \tag{2.3}$$

where $u(\cdot)$ is a (convex) utility function and the wealth of the portfolio is given by

$$W_t = \pi_t \frac{W_t}{S_t} dS_t + (1-\pi_t)\frac{W_t}{R_t} dR_t. \tag{2.4}$$

With the assumption that the parameters μ and σ are known with certainty, Merton [4, 5] obtained analytical expressions for the dynamic allocation strategy, π_t when the utility function takes certain forms. For example, when

$$u(w) = \frac{w^p}{p}, \quad p < 1, \tag{2.5}$$

the optimal allocation is given by

$$\pi_t = \frac{\mu - r}{(1-p)\sigma^2}, \tag{2.6}$$

and the value function is given as

$$J = \frac{W^p}{p} V, \tag{2.7}$$

where

$$V = \exp\left[\left(pr + \frac{p(\mu - r)^2}{2(1-p)\sigma^2}\right)(T-t)\right]. \tag{2.8}$$

2.1 Asset allocation under estimation error. During the workshop, we decided to approach the problem as follows. We assume that the risky asset follows the stochastic equation (2.1), where the return of the asset is given by

$$\mu = \mu_0 + \sigma_0 U, \tag{2.9}$$

where $U \sim N(0,1)$. Solving the asset equation (2.1) yields

$$\frac{S_t}{S_0} = \exp\left[\left(\mu_0 - \frac{\sigma^2}{2}\right) + \sigma X_t\right], \tag{2.10a}$$

where

$$X_t = aUt + B_t, \quad a = \frac{\sigma_0}{\sigma}. \tag{2.10b}$$

We now write

$$dX_t = H_t dt + K_t dZ_t, \tag{2.11a}$$

where Z_t is a Brownian motion with respect to the same filtration as the asset S_t. It can be shown that

$$H_t = a g_t X_t, \quad K_t = 1, \tag{2.11b}$$

where

$$g_t = \frac{a}{1 + a^2 t}. \tag{2.11c}$$

Thus,

$$dX_t = a g_t X_t dt + dZ_t.$$

Applying Ito's lemma, we can rewrite the process for the risky asset in terms of the observable parameters:

$$\frac{dS_t}{S_t} = (\mu_0 + \sigma a g_t X_t)dt + \sigma dZ_t. \tag{2.12}$$

Therefore, the wealth process can now be written as

$$\frac{dW_t}{W_t} = \left[r + \pi_t(\mu_0 + \sigma a g_t X_t - r)\right]dt + \pi_t \sigma dZ_t. \tag{2.13}$$

We now write the value function defined in (2.3) as $J = J(W_t, X_t, t)$, and apply Ito's lemma (note J is a martingale when we use the optimal allocation strategy) to obtain

$$dJ = (J_t + \mathcal{A}_t J)dt + J_w dW_t + J_x dX_t, \tag{2.14a}$$

with

$$\mathcal{A}_t J = \frac{1}{2} J_{xx} + \frac{\pi_t^2}{2} \sigma^2 w^2 J_{ww} + \pi_t \sigma w J_{wx}. \tag{2.14b}$$

Because J is a martingale, we have $J_t + \mathcal{A}_t J = 0$ when π_t is optimal; or

$$J_t + \frac{1}{2} J_{xx} + rwJ_w + ag_t x J_x + \frac{1}{2} \pi_t^2 \sigma^2 w^2 J_{ww} + \pi_t \left[(\mu_0 + a\sigma g_t x - r) w J_w + \sigma w J_{wx} \right] = 0, \tag{2.15}$$

with $J(w, x, T) = u(w)$. Using the constant relative risk aversion (CRRA) utility $u(w) = w^p/p$, and assuming $J = V(x,t)u(w)$, we obtain the following Hamilton-Jacobi-Bellman (HJB) equation for V:

$$V_t + \frac{1}{2} V_{xx} + ag_t x V_x + prV + \max_{\pi_t} \left\{ \frac{p(p-1)}{2} \pi_t^2 \sigma^2 V + p\pi_t \left[(\mu_0 + \sigma a g_t x - r) V + \sigma V_x \right] \right\} = 0 \tag{2.16}$$

for $p > 0$, and

$$V_t + \frac{1}{2} V_{xx} + ag_t x V_x + prV + \min_{\pi_t} \left\{ \frac{p(p-1)}{2} \pi_t^2 \sigma^2 V + p\pi_t \left[(\mu_0 + \sigma a g_t x - r) V + \sigma V_x \right] \right\} = 0 \tag{2.17}$$

for $p < 0$. The terminal condition is $V(x,T) = 1$. The allocation strategy is given by the first order condition

$$\pi_t^* = \frac{\mu_0 + a\sigma g_t x - r}{(1-p)\sigma^2} + \frac{V_x}{(1-p)\sigma V}. \tag{2.18}$$

The first order condition is only the necessary condition for optimality. Applying the second order condition yields

$$p(1-p)\sigma^2 V < 0, \quad \text{for } p > 0; \quad \text{and} \quad p(1-p)\sigma^2 V > 0, \quad \text{for } p < 0. \tag{2.19}$$

Because $V > 0$ and $p < 1$, this condition is satisfied for both positive and negative values of p.

2.2 An alternative formulation. One approach to incorporate estimation error is to treat the problem as an optimal asset allocation problem with learning. In [1], Brennan analyzes the effect of uncertainty about the mean return on the risky asset on the portfolio decision, while assuming the volatility is a known constant. To be more specific, he assumes that the change in the conditional expectation of the stock return is given by

$$dm = \frac{v_t}{\sigma^2} \left(\frac{dS}{S} - m dt \right), \tag{2.20}$$

where the conditional variance, v_t, is determined by its initial value v_0 and the differential equation

$$dv_t = -\frac{v_t^2}{\sigma^2} dt. \tag{2.21}$$

We can solve for the conditional variance to get

$$v_t = \frac{v_0 \sigma^2}{v_0 (t - t_0) + \sigma^2}. \tag{2.22}$$

Using Bellman's principle, under the optimal allocation policy, we have $\mathbb{E}[dJ] = 0$, which leads to the following Hamilton-Jacobi-Bellman (HJB) equation (after applying Ito's lemma):

$$J_t + rwJ_w + \frac{v^2}{2\sigma^2} J_{mm} + \max_{\pi_t} \left[\pi_t(m-r)wJ_w + \pi_t vwJ_{wm} + \frac{1}{2}(\pi_t \sigma w)^2 J_{ww} \right] = 0, \quad (2.23)$$

with terminal condition $J(w, m, T) = w^p/p$. If we assume $J = Vw^p/p$, (2.23) can be further simplified as

$$V_t + rpV + \frac{v^2}{2\sigma^2} V_{mm} + \max_{\pi_t} p \left[\pi_t(m-r)V + \pi_t vV_m + \frac{p-1}{2}(\pi_t \sigma)^2 V \right] = 0 \quad (2.24)$$

for $p > 0$ and

$$V_t + rpV + \frac{v^2}{2\sigma^2} V_{mm} + \min_{\pi_t} p \left[\pi_t(m-r)V + \pi_t vV_m + \frac{p-1}{2}(\pi_t \sigma)^2 V \right] = 0 \quad (2.25)$$

for $p < 0$. The terminal condition now becomes $V(m, T) = 1$.

Once again, the optimal allocation strategy is given by the first order condition

$$\pi_t^* = \frac{m-r}{(1-p)\sigma^2} + \frac{vV_m}{(1-p)\sigma^2 V}. \quad (2.26)$$

Note that π_t^* consists of two terms where the first term, denoted by π_t^m, corresponds to the ad hoc strategy in which Merton's formula is used by replacing μ with m. Note that the Merton's formula should only be applicable when there is no uncertainty, in which case the initially estimated return m_0 is used. The second term in (2.26) is the "correction" due to learning. This correction reflects how well the ad hoc strategy approximates the true strategy.

2.3 Relationship between the two formulations. We now show that the two approaches described above are equivalent. Note that the two variables m of (2.23) and x of (2.16) are related by the following equation

$$m = \mu_0 + \sigma a g_t x = m_0 + \sigma_0 g_t x. \quad (2.27)$$

Treating (2.27) as a coordinate transformation, and using simple straightforward calculations, we have the following:

$$V_t|_x = V_t|_m + V_m \frac{\partial m}{\partial t} = V_t + \sigma a g_t' x V_m, \quad (2.28a)$$

$$V_x = V_m \frac{\partial m}{\partial x} = \sigma a g_t V_m, \quad (2.28b)$$

$$V_{xx} = (\sigma a g_t)^2 V_{mm}. \quad (2.28c)$$

Substituting these equations into the HJB equation (2.16) (with π_t the optimal strategy), we obtain

$$V_t + \frac{1}{2}(\sigma a g_t)^2 V_{mm} + [\sigma a^2 g_t^2 x + p\sigma^2 a g_t \pi_t + \sigma a g_t' x] V_m$$
$$+ p \left[r + \pi_t(\mu_0 + \sigma a g_t x - r) + \frac{p-1}{2}\sigma^2 \pi_t^2 \right] V = 0. \quad (2.29)$$

Using (2.11c), we have $g' = -a g_t^2$. Noting that $v_0 = \sigma_0^2$, we obtain

$$\sigma a g_t = \frac{\sigma a^2}{a^2 t + 1} = \frac{\sigma \sigma_0^2}{\sigma_0^2 t + \sigma^2} = \frac{\sigma v_0}{v_0 t + \sigma^2} = \frac{v}{\sigma},$$

and (2.29) becomes

$$V_t + \frac{v^2}{2\sigma^2} V_{mm} + pv\pi_t V_m + p\left[r + \pi_t(m-r) + \frac{p-1}{2}\sigma^2\pi_t^2\right] V = 0, \qquad (2.30)$$

which is the same as (2.31) derived by Brennan [1] when π_t is optimal.

Remark 2.1 In [1], the HJB equation is solved directly using a finite difference method. In this report, we will present a more efficient method, which reduces the problem to solving a system of ordinary differential equations instead of the highly nonlinear partial differential HJB equation. We will also extend the model by including constraints on the allocation strategy. In practise, there are often restrictions on short selling of the risky asset as well as on the amount one can borrow. Therefore, we will impose a constraint on borrowing as well as on short-selling by considering portfolio strategies in a bounded region, i.e., $0 \le \pi_t \le 1$. Finally, learning, as well as portfolio selection, are carried out in discrete time. Therefore, it is of practical interest to study the problem under discrete time settings. We will present the continuous time approach first, and defer discussion of the discrete time approach to later in the report.

2.4 Asset allocation under estimation error and borrowing constraints. It is straightforward to implement the constraint $0 \le \pi_t \le 1$. Bellman's principle applies in a similar fashion and the simplified HJB equation can be written as

$$V_t + rpV + \frac{v^2}{2\sigma^2} V_{mm} + \max_{0\le\pi_t\le1} p\left[\pi_t(m-r)V + \pi_t v V_m + \frac{p-1}{2}(\pi_t\sigma)^2 V\right] = 0, \qquad (2.31)$$

where the value function is $J = Vw^p/p$, and the terminal condition is also the same $V(m,T) = 1$. We have implicitly assumed that $p > 0$. Otherwise, we take the minimum instead of the maximum.

The asset allocation strategy can be obtained by applying the first order condition to (2.31), which is

$$\pi_t = \max\{0, \min\{1, \pi_t^*\}\},\ \pi_t^* = \frac{m-r}{(1-p)\sigma^2} + \frac{vV_m}{(1-p)\sigma^2 V}. \qquad (2.32)$$

3 Solution methodologies

In this report we discuss two methods for solving the HJB equation: direct numerical method and a dimension reduction technique.

3.1 Numerical method for solving the Hamilton-Jacobi-Bellman (HJB) equation. For simplicity, instead of solving (2.16) or (2.17) backwards in time from terminal time T, we introduce the following change of variable:

$$s = T - t, V(x,t) \longrightarrow V(x,s), g_t \longrightarrow g_s = \frac{a}{1 + a^2(T-s)}.$$

We thus get an initial value problem for V:

$$V_s = \frac{1}{2}V_{xx} + [p\pi_s\sigma + ag_s x]\,V_x + p\max_{\pi_s}\left[r + \pi_s\left(\mu_0 + \sigma a g_s x - r\right) + \frac{(p-1)\pi_s^2\sigma^2}{2}\right], \qquad (3.1)$$

with initial condition $V(x,0) = 1$. In this case, using the first order condition, the optimal allocation strategy (unconstrained) is

$$\pi_s = \frac{\mu_0 + \sigma a g_s x - r}{(1-p)\sigma^2} + \frac{V_x}{(1-p)\sigma V}, \tag{3.2}$$

with constraints $0 \le \pi_s \le 1$.

At this juncture, we must fully specify the conditions on V for large x. Because the partial differential equation (PDE) is defined for $x \in \mathbb{R}$, we need to truncate the computational region to $x \in [-X_{max}, X_{max}]$. This requires us to impose boundary conditions at the endpoint $\pm X_{max}$. Assuming that for large x, the solution V behaves like the Merton solution $V \sim \exp(sp[r + (\mu_0 - r)^2/(2(1-p)\sigma^2])$, we shall impose the time-varying Robin conditions

$$V_x = \frac{p(x-r)s}{(1-p)\sigma^2} V, \qquad x = \pm X_{max}. \tag{3.3}$$

Even though other choices of numerical boundary conditions are possible, we will provide more insights in Section 3.2 to show that (3.3) is probably the best choice.

A few qualitative comments are in order. First, the same method can be applied to (2.24) or (2.25). Second, the constrained problem can be solved similarly. In all the cases, the PDE is highly nonlinear. However, one may naively expect that the linear diffusive term will ameliorate numerical difficulties, as long as p is much smaller than 0. This observation motivates the choice of the method of lines for discretizing the PDE.

In the x direction, we pick a mesh size $h \ll 1$, and define a uniform grid $\{x_i\}_{i=1}^N$ on $[-X_{max}, X_{max}]$. We use a centred difference scheme to evaluate both the first and second "spatial" derivatives, taking care to incorporate the Robin condition specified above. The resulting nonlinear system of ordinary differential equations (ODEs) is solved using a built-in Matlab routine.

3.2 A dimension reduction solution method. Even though we could apply the numerical method directly to the HJB equations, as in [1] for the unconstrained case, we are able to find a semi-analytic method that gives more insight into the solution behaviour. In addition it is much more efficient and avoids the problem of seeking artificial boundary conditions. Motivated by the Merton solution (2.8), we seek the solution in the following form

$$V = \exp[\alpha(t)(m-r)^2 + \beta(t)(m-r) + \gamma(t)], \tag{3.4a}$$

where α, β and γ are functions of t. It is a simply calculation to verify from (3.4a) that

$$V_t = \left[\dot{\alpha}(m-r)^2 + \dot{\beta}(m-r) + \dot{\gamma}\right] V, \tag{3.4b}$$

$$V_m = [2\alpha(m-r) + \beta]\, V, \tag{3.4c}$$

$$V_{mm} = \left[2\alpha + (2\alpha(m-r) + \beta)^2\right] V, \tag{3.4d}$$

where the dots denote the derivative with respect to time.

3.2.1 *Optimal allocation strategy.* Using (3.4a)–(3.4d), the optimal allocation becomes

$$\pi_t^* = \frac{2\alpha v + 1}{(1-p)\sigma^2}(m-r) + \frac{\beta v}{(1-p)\sigma^2}. \tag{3.5}$$

Substituting (3.4b)-(3.5) into (2.24) and rearranging the terms, we obtain

$$\dot{\alpha} + \frac{2v^2}{\sigma^2}\alpha^2 + \frac{p}{2(1-p)\sigma^2}(1+2\alpha v)^2 = 0, \tag{3.6a}$$

$$\dot{\beta} = 0, \tag{3.6b}$$

$$\dot{\gamma} + rp + \left(\frac{v}{\sigma}\right)^2 \alpha = 0, \tag{3.6c}$$

subject to

$$\alpha(T) = \beta(T) = \gamma(T) = 0. \tag{3.6d}$$

We note immediately in this case that $\beta \equiv 0$. Thus the optimal allocation strategy becomes

$$\pi_t^* = \frac{2\alpha v + 1}{(1-p)\sigma^2}(m - r), \tag{3.7}$$

and the problem at hand becomes extremely simple: in order to find the optimal allocation strategy, we only need to solve (3.6a), an *ordinary differential equation* (ODE) instead of the full HJB equation. Furthermore, the optimal asset allocation strategy is a *linear* function of m. Because the Merton solution is

$$\pi_t^m = \frac{m - r}{(1-p)\sigma^2}, \tag{3.8}$$

the difference between the current strategy and the Merton solution is also a linear function of m, i.e.,

$$\Delta\pi_t = \frac{2\alpha v}{(1-p)\sigma^2}(m - r). \tag{3.9}$$

We note that the value of γ has no effect on the allocation strategy, and its value is only needed if we want to compute the value function V (or J). The value function defined in (3.4a) becomes

$$V = \exp[\alpha(t)(m - r)^2 + \gamma(t)], \tag{3.10}$$

which takes a similar form as the original Merton solution (2.8). Furthermore, when $v \ll \sigma$ and $v \ll 1$, (3.6a) and (3.6c) can be approximated by

$$\dot{\alpha} + \frac{p}{2(1-p)\sigma^2} = 0, \tag{3.11a}$$

$$\dot{\gamma} + rp = 0. \tag{3.11b}$$

In this case we recover the Merton solution. This explains that the numerical boundary condition (3.3) is indeed a very good choice.

Finally, we note that there is a difference between the Merton strategy (where m is a constant) and the Merton's solution. If we use Merton's solution as an allocation strategy for stochastic return, we would be using the *ad hoc* strategy, and the solution of the value function could also be obtained using a similar approach, which is shown below.

3.2.2 *Solution using the ad hoc strategy.* When the ad hoc strategy

$$\pi_t^m = \frac{m - r}{(1-p)\sigma^2} \tag{3.12}$$

is used, the HJB equation (2.31) can also be simplified by substituting (3.4b)-(3.4d) and (3.12) into (2.31). In this case, we need to solve the following system of three ODEs

$$\dot{\alpha} + \frac{2v^2}{\sigma^2}\alpha^2 + \frac{1+4v\alpha}{2(1-p)\sigma^2} = 0, \tag{3.13a}$$

$$\dot{\beta} + 2\left(\frac{v}{\sigma}\right)^2 \alpha\beta + \frac{pv\beta}{(1-p)\sigma^2} = 0, \tag{3.13b}$$

$$\dot{\gamma} + rp + \left(\frac{v}{\sigma}\right)^2 \left(\alpha + \frac{\beta}{2}\right) = 0, \tag{3.13c}$$

subject to

$$\alpha(T) = \beta(T) = \gamma(T) = 0. \tag{3.13d}$$

Note that from (3.13b) and $\beta(T) = 0$, we have $\beta \equiv 0$. As a consequence, the equation for γ is the same as (3.6c). Compared with the solution using optimal allocation, the solution using the ad hoc strategy takes a similar form with minor differences, reflected in the equations for α. The difference in the equations for α in the two equations (3.6a) and (3.13a) is $2v^2\alpha^2/(1-p)\sigma^2$. Since $(v/\sigma)^2$ is normally small, the difference in the solution is also small except when α is large.

Remark 3.1 For the unconstrained case, the solution methodology used here is very effective and provides insights into the behaviour of the solution. It also gives justification for the boundary conditions when numerical methods are applied directly to the HJB equation. However, for the optimal allocation with constraints, the methodology is not applicable in general, despite the fact that similar procedure can be applied to $\pi_t = 0$ or 1 separately. It is possible that approximate solutions can be found under special circumstances, such as when $v/\sigma \ll 1$ and $v \ll 1$. Since we can solve the HJB with constraints using the finite difference method, we will not discuss it any further in this report.

4 Numerical experiments

4.1 Outline. We now describe some numerical experiments and our investigations include:

- The allocation strategy in the presence of uncertainty in the parameters, but without constraints on the allocation (Brennan's approach);
- The allocation strategy with constraints;
- A comparison of the actual allocation strategy to the "ad hoc" strategy obtained by simply replacing $\mu = m = \mu_0 + \sigma a g_t x$ in the Merton solution, i.e. using
$$\pi_t^m = \frac{m-r}{(1-p)\sigma^2} = \frac{\mu_0 + a\sigma g x - r}{(1-p)\sigma^2};$$
- Computations using the simpler dimension reduction formulation when applicable.

4.2 Results. In order to compare with the results presented by Brennan [1], in this section we make the following choices for parameters:

- Rate of return on risk-free asset $r = 5\%$;
- Volatility of the market $\sigma = 20.2\%$;
- Initial mean return of the risky asset $\mu_0 = 13\%$;
- Variance around the mean return $v_0 = 0.0243^2$;
- Risk aversion parameter $p = -2$ and $p = 0.2$;
- Time horizon $T = 5$ and $T = 20$;

- Size of the computational domain for HJB is set to be $X_{max} = 0.5$ (or $m_{max} = 0.5$ when Brennan's formulation is used).

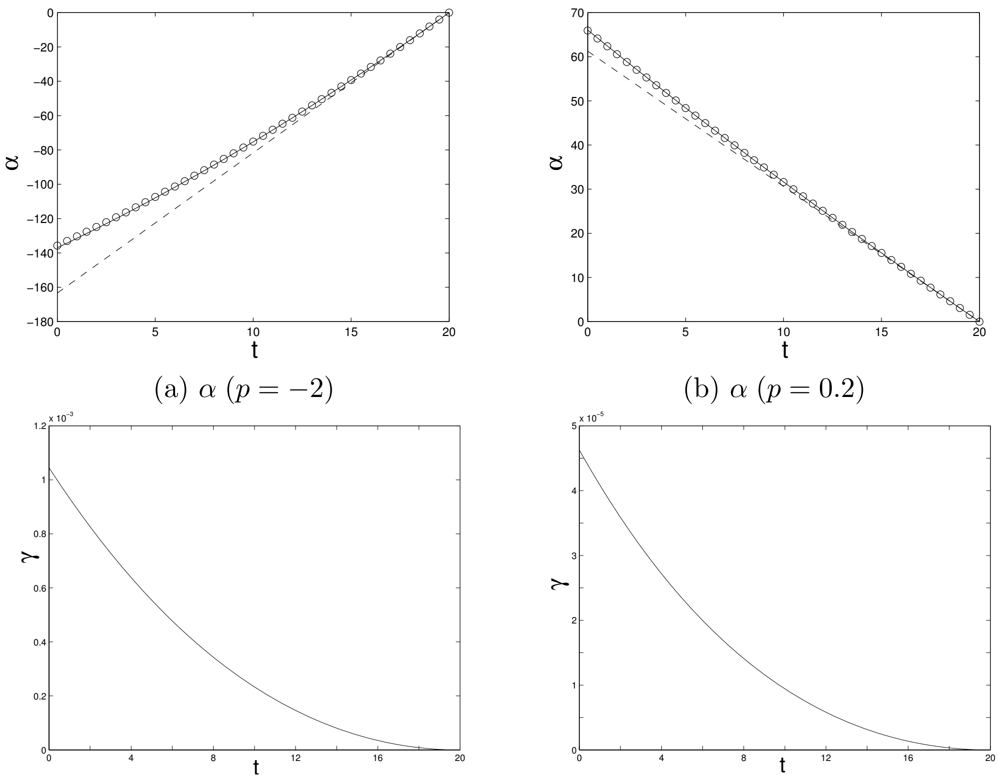
Table 1 Comparisons of the optimal allocations obtained by solving the HJB, using the dimension reduction technique as well from the case with constraint. The results from Brennan [1] are also presented.

	p	T	σ	π_t^*	π_t^m	$\pi_t^m - \pi_t^*$
HJB (2.24)	-2	5	0.202	0.6236	0.6535	0.02995
Brennan [1]				0.624	0.654	0.030
Dimension reduction (3.5)-(3.6c)				0.6235	0.6535	0.03007
HJB (2.31)				0.6236	0.6535	0.02995
HJB (2.24)	-2	20	0.202	0.5497	0.6535	0.1039
Brennan [1]				0.551	0.654	0.103
Dimension reduction (3.5)-(3.6c)				0.5478	0.6535	0.1039
HJB (2.31)				0.5497	0.6535	0.1039
HJB (2.25)	0.2	5	0.202	2.4960	2.4507	-0.0453
Brennan [1]				2.495	2.451	-0.044
Dimension reduction (3.5)-(3.6c)				2.4959	2.4507	-0.0451
HJB (2.31)				1	2.4517	1.4507
HJB (2.25)	0.2	20	0.202	2.6449	2.4507	-0.1941
Brennan [1]				2.643	2.451	-0.192
Dimension reduction (3.5)-(3.6c)				2.6419	2.4507	-0.1997
HJB (2.31)				1	2.4507	1.4507

In Table 1, we have presented the results obtained using the finite difference method for HJB equations (2.24) and (2.25), the dimension reduction method, and those from Brennan [1]. We have also presented the results for the constrained allocation case, obtained by solving the HJB equation (2.31) by finite difference method. It can be seen that all the results for the unconstrained case agree with each other. It is interesting to note that when the constraint is not active, the values of the optimal allocation stay unchanged. As pointed by in [1], the correction to the ad hoc strategy is positive for $p > 0$ and negative for $p < 0$, under economic viable conditions. This can also be explained by the sign of α in the dimension reduction solution. Because the correction is given by

$$\frac{2\alpha v}{(1-p)\sigma^2}(m-r),$$

and $v > 0$ and $p < 1$, the sign of the correction is determined by the sign of α for $m > r$ ("economically viable"). The sign of α, on the other hand, is the same as p, which can be seen clearly from (3.6a), when $v/\sigma \ll 1$, as is the case here. In Figure 1, the numerically computed values of α and γ are given for the optimal allocation (both exact and approximated) and the ad hoc allocation strategies. We can see from the graphs that the difference between the optimal and ad hoc strategies is small. The approximation (Merton's solution) is reasonably close to the exact value of α. We also present the numerically computed value functions in Figure 2. Finally, the optimal allocation π_t^* at $t = 0$ is given in Figure 3. Linear variation with m is apparent in all cases, even for the constrained problem when the constraints are not active.



(a) α ($p = -2$) (b) α ($p = 0.2$)

(c) γ ($p = -2$) (d) γ ($p = 0.2$)

Figure 1 Numerically computed values of α and γ, using (3.6a) and (3.6c). The solid line corresponds to the optimal allocation and the circles are for the ad hoc allocation. The dashed line is from the approximation of the optimal allocation (Merton's solution).

5 Discrete time model

In this section we describe a time-discrete formulation of the asset allocation problem with learning. This is a more realistic approach in the sense that both market transactions and learning occur in discrete time intervals (rather than continuously).

5.1 Illustration A: one period. Consider a market consisting of one risky asset S_t and a bond R_t. The prices can be written as (with respect to the 'full' sigma field $\mathcal{G}_t$)

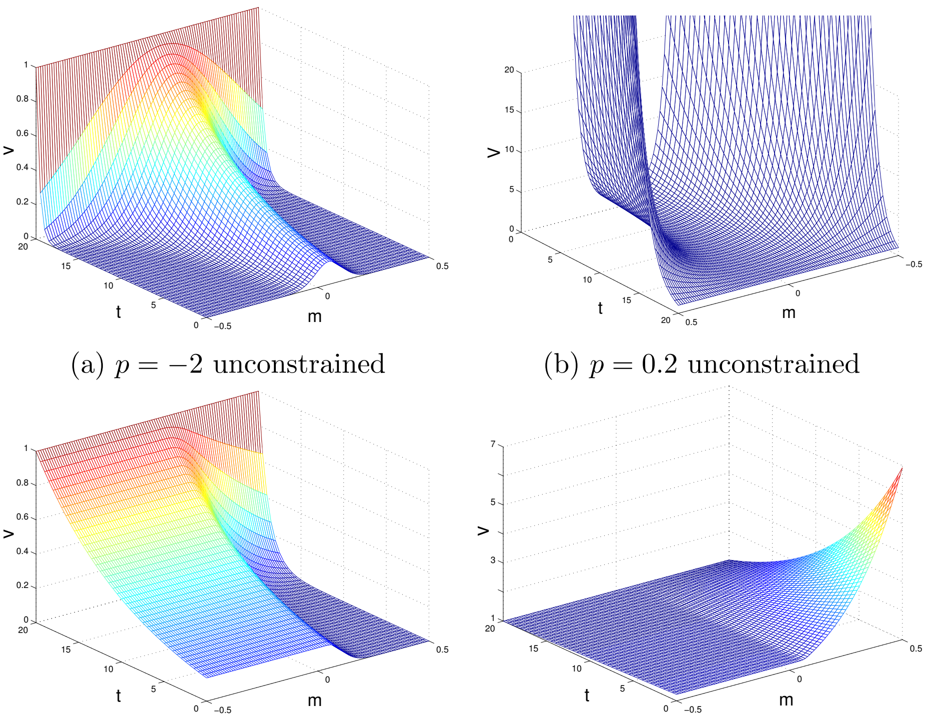
$$\begin{cases} \mathrm{d}S_t &= \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}Z_t \\ \mathrm{d}R_t &= rR_t \mathrm{d}t \end{cases} \Rightarrow \begin{cases} S_t &= S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma Z_t} \\ R_t &= R_0 e^{rt} \end{cases}$$

where we take μ to be Gaussian with mean μ_0 and standard deviation σ_0. With respect to the sigma field generated by the market up to time t_0, $\mathcal{F}_{t_0} = \sigma\{S_k, k < t_0\}$, the price of the stock can then be written

$$S_t = S_{t_0} e^{(\mu_0 - \frac{1}{2}\sigma^2)(t - t_0) + \sigma(X_t - X_{t_0})},$$

where

$$X_t = X_0 + \frac{\sigma_0}{\sigma} tU + Z_t, \quad U \sim \mathcal{U}(0, 1).$$



(a) $p = -2$ unconstrained (b) $p = 0.2$ unconstrained

(c) $p = -2$ constrained (d) $p = 0.2$ constrained

Figure 2 Numerically computed value function V, using (2.24) and (2.25).

We now consider the discrete time one period portfolio allocation problem. Let us assume $t_0 = 0$ and $(t - t_0) = 1$. The total wealth after investing in the market for one period with a strategy given by the *starting* strategy π_0 will be

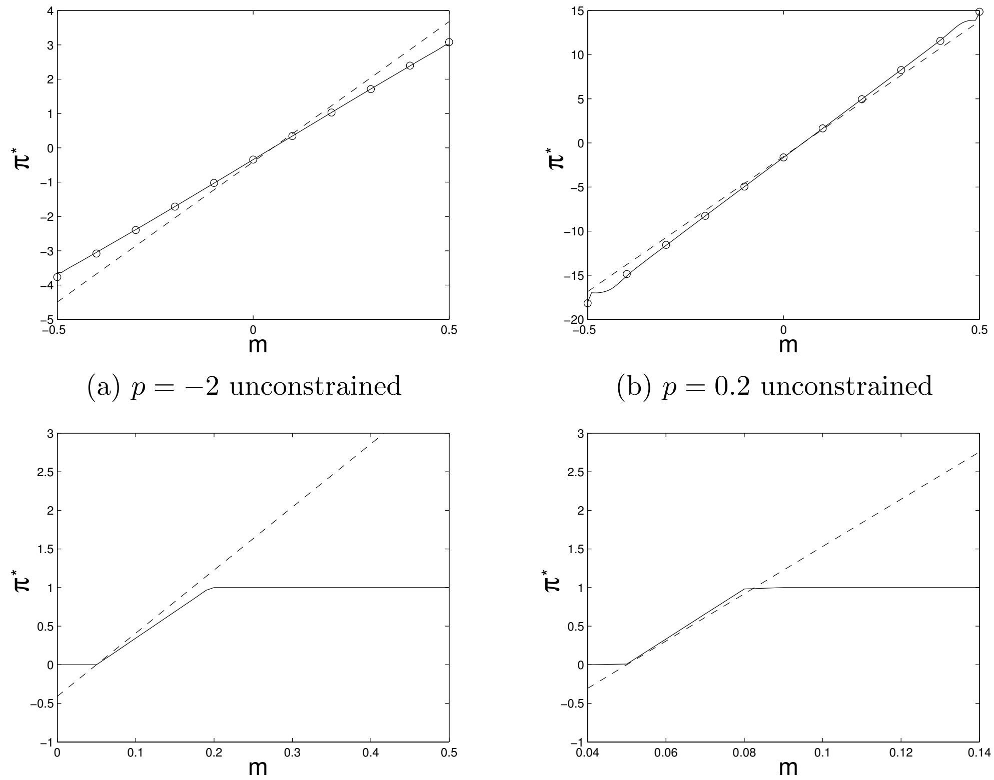
$$W_1 = W_0 \left(\pi_0 \frac{S_1}{S_0} + (1 - \pi_0) \frac{R_1}{R_0} \right).$$

The goal of the investor is to maximize the expectation of the utility of wealth

$$V(\pi_0, W_0, t_0) = \mathbb{E}(U(W_1)|\mathcal{F}_0).$$

If we take a linear utility function, $U(x) = x$, we can easily calculate the expectation and solve the maximizing problem explicitly:

$$\begin{aligned} V(\pi_0, W_0, t_0) &= \mathbb{E}(W_1|\mathcal{F}_0) \\ &= W_0 \mathbb{E} \left(\pi_0 e^{\mu_0 - \frac{1}{2}\sigma^2 + \sigma(X_1 - X_0)} + (1 - \pi_0) e^r \right) \\ &= W_0 \left(\pi_0 \left(e^{\mu_0 + \frac{1}{2}\sigma_0{}^2} - e^r \right) + e^r \right) \\ &= W_0 \frac{1}{2\pi} \int \left[\pi \exp \left(\mu_0 - \frac{1}{2}\sigma^2 - \frac{-X^2 - 2\hat{\sigma}^2 X + \hat{\sigma}^4}{2\hat{\sigma}^2} + \frac{\hat{\sigma}^2}{2} \right) \right. \\ &\qquad \left. + (1 - \pi) \exp(r - \frac{X^2}{2\hat{\sigma}^2}) \right] dX. \end{aligned}$$



(a) $p = -2$ unconstrained (b) $p = 0.2$ unconstrained

(c) $p = -2$ constrained (d) $p = 0.2$ constrained

Figure 3 Numerically computed values of optimal allocation π_t^* at $t = 0$. The solid line is the solution by solving the HJB equation using finite difference method, the dashed line is the ad hoc strategy, and the circles are the solution using the dimension reduction method.

Here X is a normally distributed random variable with mean zero and standard deviation $\hat{\sigma} = \sqrt{\sigma^2 + \sigma_0^2}$. V is maximized with the allocation strategy

$$\pi_0 = \begin{cases} 0 & \text{if } \mu_0 + \frac{1}{2}\sigma_0^2 < r, \\ 1 & \text{if } \mu_0 + \frac{1}{2}\sigma_0^2 > r, \end{cases}$$

and the investor is indifferent if $\mu_0 + \sigma_0^2/2 = r$. In other words, the allocation strategy is determined by the relation of the uncertainty in the market parameter μ, and *not* on the standard deviation σ of the stock. This is to be contrasted to the result for the optimal allocation problem with deterministic μ, where the optimal strategy π_0 is

$$\pi_0 = \begin{cases} 0 & \text{if } \mu + \frac{1}{2}\sigma^2 < r, \\ 1 & \text{if } \mu + \frac{1}{2}\sigma^2 > r. \end{cases}$$

5.2 Illustration B: two periods. Now consider investing in the same market for two investment periods. The investor must make two allocation decisions represented by π_0 and

π_1 (at time 0 and 1 respectively). The problem facing the investor can be phrased

$$\max_{(\pi_0,\pi_1)} \mathbb{E}\left(U(W_2)|\mathcal{F}_0\right), \qquad W_2 = W_1\left(\pi_1\frac{S_2}{S_1} + (1-\pi_1)\frac{R_2}{R_1}\right).$$

The expectation can be rewritten

$$\begin{aligned}
&\mathbb{E}\left(\mathbb{E}\left(W_2|\mathcal{F}_1\right)|\mathcal{F}_0\right)\\
&\quad = \mathbb{E}\left[\mathbb{E}\left[W_0\left(\pi_0\frac{S_1}{S_0} + (1-\pi_0)\frac{R_1}{R_0}\right)\left(\pi_1\frac{S_2}{S_1} + (1-\pi_1)\frac{R_2}{R_1}\right)\Big|\mathcal{F}_1\right]\Big|\mathcal{F}_0\right]\\
&\quad = \mathbb{E}\left[W_0\left(\pi_0\frac{S_1}{S_0} + (1-\pi_0)\frac{R_1}{R_0}\right)\left(\pi_1\mathbb{E}(\frac{S_2}{S_1}|\mathcal{F}_1) + (1-\pi_1)\frac{R_2}{R_1}\right)\Big|\mathcal{F}_0\right]\\
&\quad = W_0\pi_0\mathbb{E}(\pi_1\frac{S_1}{S_0}\mathbb{E}(\frac{S_2}{S_1}|\mathcal{F}_1)|\mathcal{F}_0) + W_0\pi_0\frac{R_2}{R_1}\mathbb{E}((1-\pi_1)\frac{S_1}{S_0}|\mathcal{F}_0)\\
&\qquad + W_0(1-\pi_0)\frac{R_1}{R_0}\mathbb{E}(\pi_1\mathbb{E}(\frac{S_2}{S_1}|\mathcal{F}_1)|\mathcal{F}_0) + W_0(1-\pi_0)\frac{R_2}{R_0}\mathbb{E}((1-\pi_1)|\mathcal{F}_0).
\end{aligned}$$

Writing out S_1 and S_2 explicitly in the above expression, we see that we only need to calculate

$$\mathbb{E}(\pi_1\frac{S_1}{S_0}\mathbb{E}(\frac{S_2}{S_1}|\mathcal{F}_1)|\mathcal{F}_0), \tag{5.1}$$

$$\mathbb{E}((1-\pi_1)\frac{S_1}{S_0}|\mathcal{F}_0), \tag{5.2}$$

$$\mathbb{E}(\pi_1\mathbb{E}(\frac{S_2}{S_1}|\mathcal{F}_1)|\mathcal{F}_0), \tag{5.3}$$

$$\mathbb{E}\left[1-\pi_1|\mathcal{F}_0\right]. \tag{5.4}$$

Note that π_1 depends on the market σ field $\mathcal{F}_1$ (i.e. $\pi_1 = \pi_1(S_1)$) in an, as yet, undetermined way. To simplify these expectations as much as possible we note that with respect to $\mathcal{F}_{t_0}$, we have

$$\mathrm{d}X_t = ag_tX_t\mathrm{d}t + \mathrm{d}Z_t, a = \frac{\sigma_0}{\sigma}, \quad g_t = \frac{a}{a^2t+1}.$$

Thus, X_t with respect to $\mathcal{F}_{t_0}$ is an Ornstein-Uhlenbeck process, and

$$X_t \sim \mathcal{N}(\mu_x(t_0,t), \sigma_x(t_0,t)^2) = \mathcal{N}\left(e^{\int_{t_0}^t ag_s\mathrm{d}s}X_{t_0}, \int_{t_0}^t e^{2\int_{t'}^t ag_s\mathrm{d}s}\mathrm{d}t'\right). \tag{5.5}$$

We then have

$$\begin{aligned}
\mathbb{E}\left[\frac{S_2}{S_1}|\mathcal{F}_1\right] &= e^{\mu_0-\frac{1}{2}\sigma^2-\sigma X_1}\mathbb{E}\left[e^{\sigma X_2}|\mathcal{F}_1\right]\\
&= e^{\mu_0-\frac{1}{2}\sigma^2+\sigma\mu_x(1,2)+\frac{1}{2}\sigma^2\sigma_x(1,2)^2-\sigma X_1}.
\end{aligned}$$

The expectations (5.1), (5.2), (5.3), (5.4) then simplify to

$$\begin{aligned}
\mathbb{E}(\pi_1\frac{S_1}{S_0}\mathbb{E}(\frac{S_2}{S_1}|\mathcal{F}_1)|\mathcal{F}_0) &= \mathbb{E}(\pi_1|\mathcal{F}_0)e^{2(\mu_0-\frac{1}{2}\sigma^2)+\sigma\mu_x(1,2)+\frac{1}{2}\sigma^2\sigma_x(1,2)^2-\sigma X_0}, \\
\mathbb{E}((1-\pi_1)\frac{S_1}{S_0}|\mathcal{F}_0) &= \mathbb{E}((1-\pi_1)e^{\sigma X_1}|\mathcal{F}_0)e^{\mu_0-\frac{1}{2}\sigma^2-\sigma X_0}, \\
\mathbb{E}(\pi_1\mathbb{E}(\frac{S_2}{S_1}|\mathcal{F}_1)|\mathcal{F}_0) &= \mathbb{E}(\pi_1 e^{-\sigma X_1}|\mathcal{F}_0)e^{\mu_0-\frac{1}{2}\sigma^2+\sigma\mu_x(1,2)+\frac{1}{2}\sigma^2\sigma_x(1,2)^2}, \\
\mathbb{E}\left[1-\pi_1|\mathcal{F}_0\right] &= 1-\mathbb{E}\left[\pi_1|\mathcal{F}_0\right].
\end{aligned}$$

In principle one could now attempt to solve the constrained optimization problem by setting up the Lagrangian and differentiating with respect to $\{\pi_1\}$, being careful about switching the order of expectation and differentiation. That is for each s we would need to set the derivative of the expected utility of wealth (E.U.W.) to zero. However, we can save a lot of effort by noting that the E.U.W. depends only linearly on $\{\pi_1\}$. This means that, once again, the optimal solution for each path $s = \{S_t, 0 \le t \le 1\}$ is either zero or one. Writing the E.U.W. as

$$\text{E.U.W.} = \mathbb{E}(A(X_1) + B(X_1)\pi_1(s)|\mathcal{F}_0),$$

we see that the optimal π_1 only depends on X_1. Since π_1 can only depend on information derived from observables (S_t), we must build a 'best inference' of X_1, say $\hat{X}_1(s)$, in order to estimate the sign of $B(X_1)$. Then the optimal investment decision for the second period would be

$$\pi_1^* = \begin{cases} 0 & \text{if } B(\hat{X}_1(s)) > 0, \\ 1 & \text{if } B(\hat{X}_1(s)) <= 0. \end{cases}$$

5.3 A non-linear utility function. In researching the best possible utility function that describes investment portfolio strategies, a concave function of the form $u(W) = -e^{\frac{1}{\gamma}W_0}, \gamma > 0$, was found to capture the required behaviour [8] (p.419).

The solution to the one-step discrete model is already analytically not possible without making some approximations. The solution process is the following:

$$\begin{aligned}
\max_{\{\pi\}} E[u(W_0)|\pi] &= const \int e^{-W_0\frac{1}{\gamma}\pi e^{\mu-\frac{\sigma^2}{2}+\sigma Z}} e^{-W_0\frac{1}{\gamma}(1-\pi)e^r} e^{-\frac{z^2}{2}} dz \\
&\approx -\frac{1}{\gamma}e^{-\frac{1}{\gamma}(1-\pi)e^r}\pi e^{mu-\frac{1}{2}\sigma^2+\frac{\sigma^2}{2}} + \frac{1}{\gamma^2}e^{-\frac{1}{\gamma}(1-\pi)e^r}\pi^2 e^{2\mu-\frac{\sigma^4}{4}+\sigma^2}.
\end{aligned}$$

When the above is provided as input in Maple, the solution obtained is of the form

$$-\frac{1}{2}\gamma\left(-e^r + 2e^{\mu-\sigma^4/4+\sigma^2} - \sqrt{e^{2r} + 4e^{2\mu-\sigma^4/2+2\sigma^2}}\right)e^{-\mu-r+\sigma^4/4-\sigma^2}.$$

6 Conclusion

In this report, we investigate the impact of uncertainty in the market parameters to the optimal allocation problem. An alternative derivation to the one proposed by Brennan is presented. A dimension reduction solution is obtained by reducing the HJB equation into a system of simpler ODEs. Numerical results demonstrate the validity of these approaches, and the impact of the constraints on the allocation strategy is discussed. Discrete-time allocation models are also presented. To simplify the discussion, we focus on asset allocation and do not consider consumption. The methodology that we present in this report can be extended to include consumption, which will be pursued in a future study.

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