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Stability of Stationary Velocity Profiles in Fiber Spinning

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1. The problem

We consider a process of fiber spinning, where a viscous (but not necessarily Newtonian) fluid is being pushed through a narrow "spinning hole" and, upon exit, is being stretched, the latter step in order to obtain an acceptable alignment of the (polymeric) molecules in the fluid. This alignment is necessary for the final mechanical properties of the fiber. After leaving the spinning hole, the fluid passes through a layer of air (the "air gap") and enters a bath, in which it solidifies almost instantaneously. Somewhere inside this bath, the fiber is being drawn by a wheel, which delivers the force, necessary for the stretching process.
When the speed of the drawing wheel is set to high, it is impossible to obtain a uniform fiber: one clearly observes variations in the fiber diameter. This phenomenon is called *draw resonance*. Experimental evidence suggests that the *draw ratio*, that is, the ratio between the speeds at the wheel and at the exit of the spinning hole, is the unique parameter to steer the onset of draw resonance, and for Newtonian fluids this is known to be true. The reader is referred to [1] and [2]. In the Newtonian case, the onset of draw resonance can be shown to be a Hopf bifurcation.

The question, asked to the Study Group, was to extend the results on Newtonian fluids to fluids with more general rheologies, with power law fluids as a first choice. What came out of the Study Group was not yet a final solution to this problem, but rather an attempt to come to an easier formulation of the model equations. At the workshop, we thought that we had succeeded, but afterwards we found that there was a hidden mistake, which we could not easily correct. This mistake in itself is worth mentioning, because the approach we tried may be successful in other cases, and this mistake may easily slip into the considerations there. In this note, we restrict ourselves to the Newtonian case, because it is easy to describe the mistake in this setting, and the power law setting would not add anything.

As a final remark, we note that the "power law case" has been solved now, by using similar methods as we did in the Newtonian case. A publication is in preparation.

Apart from the authors, during the study group the following persons have worked on this problem: Pieter de Groen (Free University, Bruxelles) and Sandro Merino (University of Strathclyde, Glasgow).

2. Summary of the previous results

When the (very thin) fluid jet is considered one dimensional, a nondimensionalized model for the Newtonian case is given by

$$\begin{aligned} \rho_\tau + (\rho v)_x &= 0 && (1)\\ Re(\rho v_\tau + \rho v v_x) &= (\rho v_x)_x && (2)\\ \rho(0,\tau) = v(0,\tau) &= 1 && (3)\\ v(1,\tau) &= s, && (4) \end{aligned}$$

where $\tau,\ x,\ \rho$ and v are nondimensionalized time, length along the jet, cross-section and speed, respectively. The parameter s is the draw ratio. Re is the Reynolds number, defined by $Re = \frac{v_s \ell \rho}{\eta}$, where $v_s,\ \ell,\ \rho$ and η are, respectively, the speed at the exit of the spinning hole, the length of the air gap, the density and the (Trouton) viscosity. Actually, Re is very small, so that we are left with

$$(P_1)\begin{cases} \rho_\tau + (\rho v)_x = 0 \\ v_{xx} + \frac{\rho_x v_x}{\rho} = 0 \\ \rho(0,\tau) = v(0,\tau) = 1 \\ v(1,\tau) = s \\ \rho(x,0) = \hat{\rho}(x), \end{cases}$$

where $\hat{\rho}$ is an initial situation. Note that (P_1) has a unique stationary solution, given by $\rho(x,t) = \rho_0(x) = s^{-x},\ \ v(x,t) = v_0(x) = s^x$, where we have assumed that $\hat{\rho} = \rho_0$. We are interested in the *stability* of this stationary solution. Therefore, we linearize around ρ_0, v_0, as follows: we set

$$\rho = s^{-x}(1 + p(x,\tau)); \quad v = s^x(1 + q(x,\tau)),$$

and we omit all terms that are of order > 1 in $\{p,q\}$. Next, we require that the boundary condition are not perturbed: $p(0,\tau) = q(0,\tau) = q(1,\tau) = 0$. The resulting system of equations for $\{p,q\}$ can be transformed into one single equation

$$\sigma(\tau) = \frac{s \log s}{s-1} \int_0^{(s-1)/(s \log s)} \left\{ \frac{ys \log s + 1 - s}{s(1 - y \log s)^2} \right\} \sigma(\tau - y)\, dy, \tag{5}$$

where $\sigma(\tau)$ is the perturbation in drawing force. The next step is to show that the stability question is completely determined by the complex valued eigenvalues λ, given by

$$\frac{s \log s}{s-1} \int_0^{(s-1)/(s \log s)} \left\{ \frac{ys \log s + 1 - s}{s(1 - y \log s)^2} \right\} e^{-\lambda y}\, dy = 1.$$

The stationary solutions are unstable when there is an eigenvalue with positive real part, and stable when all eigenvalues are in the left half plane. A proof is contained in [3, Section I, Th.5.4].

3. Approach of the Study Group

The first purpose has been to come to a different problem formulation, which we did for the Newtonian case first. We did not a priori assume Re to be small, but we have used the letter ε in stead of Re.

We introduce the new coordinates

$$\psi(x, \tau) = \int_0^x \rho(\xi, \tau)\, d\xi, \qquad t(x, \tau) = \tau.$$

This choice is justified by the fact that $\psi_x = \rho > 0$. Note that ψ is a stream function and that

$$\psi_\tau = 1 - \rho v.$$

By abuse of notation, we use the same letters for the dependent variables as before. Upon the transformation, the region $\{0 < x < 1\}$ is transformed into the region

$$0 < \psi < \int_0^1 \rho(\xi, t)\, d\xi = \zeta(t),$$

where

$$\zeta'(t) = 1 - \rho(\zeta(t), t)s.$$

Apparently, the coordinate transformation has left us with a free boundary problem. Note that, in the stationary case, we have that $\rho(\zeta(t), t) = \frac{1}{v(\zeta(t), t)} = \frac{1}{s}$, so that $\zeta'(t) = 0$, as expected.

In the new coordinates, the differential equations read

$$\rho_t + \rho_\psi + \rho^2 v_\psi = 0, \tag{6}$$

$$v_t + v_\psi = \frac{1}{\varepsilon}(\rho^2 v_\psi)_\psi. \tag{7}$$

The equations are valid in the region $0 < \psi < \zeta(t)$, where $\zeta'(t) = 1 - \rho(\zeta(t), t)s$. As boundary conditions we have

$$\rho = v = 1 \qquad (\psi = 0), \tag{8}$$

$$v = s \qquad (\psi = \zeta(t)). \tag{9}$$

The stationary solution $\{\rho_0, v_0\}$ satisfies

$$\rho_0' + \rho_0^2 v_0' = 0, \tag{10}$$

$$\varepsilon v_0' = \left(\frac{v_o'}{v_0^2}\right)'. \tag{11}$$

By (8), we find from (10) that

$$v_0 \rho_0 = 1. \tag{12}$$

Equation (11) may be integrated, to yield that

$$\varepsilon v_0 = \frac{v_0'}{v_0^2} - c. \tag{13}$$

From this result we deduce that

$$\int_1^s \frac{dv}{v^2(\varepsilon v + c)} = \zeta_0. \tag{14}$$

In order to determine c, we recall that, in the "old" variables, we have

$$\frac{d\psi}{dx} = \rho_0 = \frac{1}{v_0},$$

so that

$$1 = \int_0^1 v_0 \frac{d\psi}{dx}\ dx = \int_0^{\zeta_0} v_0\ d\psi.$$

From (13) we deduce that $d\psi = \frac{dv_0}{v_0^2(\varepsilon v_0 + c)}$, so that

$$\int_1^s \frac{dv}{v(\varepsilon v + c)} = 1. \tag{15}$$

This relation enables us to determine c and, thus, v_0 and ρ_0.

So far so good. In order to analyze the stabilty of the steady solution, we linearise around (6), (7), writing

$$\rho = \rho_0 + u; \qquad v = v_0 + w.$$

Formally, this gives, to first order,

$$u_t + u_\psi + 2\rho_0 u v_0' + \rho_0^2 w_\psi = 0,$$

$$w_t + w_\psi = \frac{1}{\varepsilon}(2\rho_0 u v_0' + \rho_0^2 w_\psi)_\psi.$$

But here we made the following mistake. When we perturb ρ, we implicitly perturb our coordinate ψ, which must be taken into account in the linearization. We did not perceive this during the workshop, and afterwards we found that correcting the mistake made the equations rather awful.
After the workshop and the detection of the mistake, we again tried the "old" approach on power law fluids. We seem to have been successful now, and another publication is in preparation.

References

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3. Diekmann, O., S.A. van Gils, S.M. Verduyn Lunel & H.-O.Walther, *Delay Equations*, Springer, 1995.