DANISCO: Temperature and Moisture Gradients in Sugar Silos

ANDREW LACEY AND MADS PETER SØRENSEN

1 Introduction

Danisco Sugar is the main sugar producer in Denmark. The sugar is extracted from sugar beets which are harvested during the beet campaign September – December. The refined sugar is stored in huge silos and withdrawn on a regular, but not constant, basis for distribution until the next campaign. Danisco Sugar utilizes several types of silos for the storage of crystalline sugar. They differ in construction material e.g. concrete or steel plate silos, dimension, design with or without a central tower and isolation. Various methods of filling and emptying are utilized, that is first in – last out, or first in – first out. The silos are equipped with one or more facilities for sugar conditioning. Those includes temperature control of walls, control of the condition of the air space above sugar, ability to blow conditioned air through the bulk sugar etc.

A number of problems arise when storing sugar for long periods. Due to distribution of heat and moisture the sugar can 'cake' or harden and moist sugar is difficult to empty from the silos. The chemical and microbial stability of the sugar is influenced by the temperature and moisture distribution. On the other hand under very dry conditions there is a risk of dust explosions. The object is to control the temperature and moisture content within a lower and an upper limit. Danisco Development Centre wishes to employ mathematical modelling as a tool for understanding and predicting temperature and moisture distribution in sugar stored in silos.

In moisture migration the physical processes which have to be taken into account are diffusion in the porous bulk sugar, natural convection due to temperature gradients, forced convection due to bulk sugar air conditioning, conditioning of the air above the bulk sugar and migration of moisture through concrete silo walls. The temperature distribution is influenced by seasonal variations, the temperature of the air space above the bulk sugar, remixing processes when filling and emptying silos, air conditioning and others. The following physical properties are assumed of importance when formulating models to describe the formation of temperature and moisture gradients: bulk sugar heat capacity, heat conductivity, density, porosity, tortuosity and sorption isotherms. Most will vary with the sugar quality (that is: crystal size, reducing sugar content and ash content), some with temperature and moisture.

Mathematical models for moisture and heat conduction in stored cereal grains have previously been developed [1, 2]. In modelling moisture and heat conduction in sugar our starting point is to use the approach applied in modelling moisture and heat conduction in grain and adopt the model to the case of stored sugar, including the possibility of air conditioning.

In our modelling carried out during the week of the Study Group, we used a number of simplifying assumptions, such as air flow through the sugar being dominated by forced, and constant, convection, and having processes happening fast enough, so that local equilibrium holds. (Estimates obtained at the time indicated that the temperature varies very little over the length scale of the sugar grains and their interstices.)

2 Sorption Isotherms

In the sugar bulk water can either be adsorbed onto the sugar crystalline in form of liquid water (moisture) or exist as water vapour in the interstitial air. The latent heat consumed during evaporation of moisture from the sugar to the air is denoted by ℓ (J/kg of H_2O). The moisture content W_s in the sugar is defined as the mass of moisture divided by the mass of dry sugar. The relative humidity h of the interstitial air is defined as the fraction p_v/p_s between the partial pressure of water vapour p_v and the saturation pressure of water vapour p_s. In equilibrium the relative humidity and the the moisture content W_s of sugar are related through a sorption isotherm at fixed temperature T. In mathematical terms we can express this relation in the form

$$\frac{p_v}{p_s} = f(W_s, T) \ , \tag{1}$$

where f is a function depending on the moisture content and temperature. The sorption isotherms are determined from experimental data and are normally presented as graphs showing W_s as function of the relative humidity at different temperatures. A typical example is displayed in Fig.1. We assume in the present model that the air-vapour mixture behaves as an ideal gas. The ideal gas law for vapour reads

$$p_v = \rho_v R_v T \ , \tag{2}$$

where p_v is the partial pressure of water vapour in air. The density of vapour is denoted ρ_v and R_v is the vapour gas constant in units of J/(kg °K). Here R_v is related to the universal gas constant $\hat{R} = 8317$ J/(kmol °K) through $\hat{R} = \hat{M}_v R_v$ where $\hat{M}_v$ denotes the molar mass of vapour. A similar expression holds for the partial pressure of air. The absolute humidity W_a is defined as

$$W_a = \frac{m_v}{m_a} = \frac{\rho_v}{\rho_a} \; . \tag{3}$$

The masses m_v and m_a are the masses of vapour and dry air, respectively, in a fixed volume.

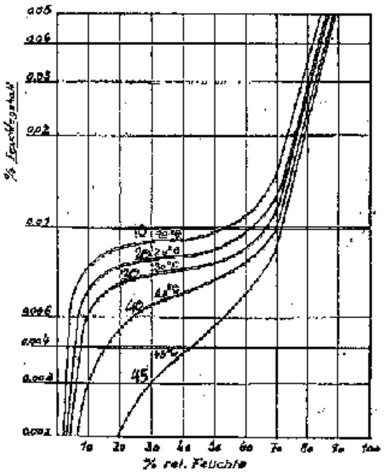

Figure 1: Sorption isotherms for equilibrium moisture relationship for sugar.

The sugar bulk is a porous medium which contains sugar crystals, liquid water, water vapour and air. The volume fraction of sugar is denoted ϕ_s. This means that the mass of sugar in a volume V_0 of the bulk is $m_s = \rho_s \phi_s V_0$, where ρ_s is the density of sugar. Similarly the volume fractions of water, air and vapour are ϕ_w, ϕ_a and ϕ_v, respectively. However, the volume fraction of air and vapour is fixed, and equals the porosity ε of the bulk; the fraction ϕ_w is believed to be normally very small.

With the above notation we can write the absolute moisture content of sugar in mathematical terms as

$$W_s = \frac{m_w}{m_s} = \frac{\rho_w \phi_w}{\rho_s \phi_s} \; , \tag{4}$$

where m_w is the mass of moisture within the sugar in a given volume V_0 of the bulk and m_s is the mass of sugar in V_0.

3 The Heat Equation

In the derivation of the equations describing heat conduction and moisture conduction, we shall assume thermodynamic equilibrium between the wet sugar and the surrounding air in the porous media. This implies that the temperature in a point of the bulk is the same for all phases and that the sorption isotherms can be used. Air flowing through the bulk sugar in order to dry or wet the sugar flows with a velocity $\mathbf{u}$ (averaged over a total volume, and not just the pores) and gives rise to convective transport of heat and moisture. The flow velocity is assumed here to be constant. We shall assume $\mathbf{u}$ to be sufficiently small so that the thermodynamic equilibrium is satisfied even in the convective case. This means that the system must reach equilibrium from an out of equilibrium case much faster than moisture is transported due to convection. With these assumptions the heat equation now reads

$$\phi_s \rho_s C_s \frac{\partial T}{\partial t} + \rho_w C_w \frac{\partial(\phi_w T)}{\partial t} + \varepsilon C_a \frac{\partial(\rho_a T)}{\partial t} + \varepsilon C_v \frac{\partial(\rho_v T)}{\partial t} \\ + \nabla \cdot (\varepsilon \rho_a C_a T \mathbf{u}) + \nabla \cdot (\varepsilon \rho_v C_v T \mathbf{u}) = \nabla \cdot (k \nabla T) + \ell \frac{\partial(\phi_w \rho_w)}{\partial t} \,. \quad (5)$$

The constants C_s, C_w, C_a and C_v are the specific heat of sugar, water, air and vapour, respectively. The constant k is the thermal conductivity of the porous bulk and it arises as an average of thermal conductivity over all the phases appearing in the bulk sugar. In the above equation we have used that the volume fractions of air and vapour are identical and equal to the porosity ε. Compared to sugar and water the heat capacity and volume fraction of air and vapour are so small that we shall neglect those terms in the first line of Eq.(5). However, we keep the convective terms. Invoking Eq.(4) the heat equation reduces to

$$\phi_s \rho_s C_s \frac{\partial T}{\partial t} + \phi_s \rho_s C_w \frac{\partial(W_s T)}{\partial t} + \varepsilon C_a \nabla \cdot (\rho_a T \mathbf{u}) + \varepsilon C_v \nabla \cdot (\rho_v T \mathbf{u}) \\ = \nabla \cdot (k \nabla T) + \ell \phi_s \rho_s \frac{\partial W_s}{\partial t} \,. \quad (6)$$

This equation involves four dependent variables, namely temperature T, the absolute moisture content in sugar W_s, and the air and vapour densities. From the ideal gas law these densities can be related to the partial presures of air and vapour. The partial presures give the relative humidity from which one can calculate the moisture content W_a in air. This is the reason why we need not involve W_a in the dynamical equations.

4 Moisture Migration

The moisture in the sugar does not migrate. The moisture migration takes place by evaporation and water-vapour diffusion through the air pores and by forced convection. Thus in the diffusion and convection equation for moisture migration, evaporation of water from the sugar acts as a source term and accordingly we have

$$\frac{\partial(\phi_v \rho_v)}{\partial t} + \nabla \cdot (\phi_v \rho_v \mathbf{u}) = \nabla \cdot \left(\frac{D_v}{\tau} \nabla(\phi_v \rho_v) \right) - \frac{\partial(\phi_w \rho_w)}{\partial t} \, . \tag{7}$$

The parameter D_v is the diffusion constant of water vapour in air and τ is the tortuosity factor of the porous media. Using the fact that the volume fraction of vapour ϕ_v equals the porosity ε and invoking Eq.(3) together with the ideal gas law we get

$$\phi_v \rho_v = \varepsilon \rho_a W_a = \varepsilon \frac{p_v}{R_v T} \, . \tag{8}$$

Inserting this expression into Eq.(7), neglecting the first term and using Eq.(4) leads to

$$\phi_s \rho_s \frac{\partial W_s}{\partial t} + \varepsilon \nabla \cdot \left(\frac{p_v}{R_v T} \mathbf{u} \right) = \varepsilon \nabla \cdot \left(\frac{D_v}{\tau} \nabla \left(\frac{p_v}{R_v T} \right) \right) \, . \tag{9}$$

In order to use the ideal gas law above, we assume local thermodynamic equilibrium of the water vapour. This may not always be a sufficiently good approximation. The final equation needed is the continuity equation for the air flowing in the pores. This equation reads

$$\frac{\partial \rho_a}{\partial t} + \nabla \cdot (\rho_a \mathbf{u}) = 0 \, . \tag{10}$$

In the previous section we mentioned that the system is described by four dependent variables. These dependent variables are governed by the heat equation (6), the moisture migration equation (9), the continuity equation (10) and the sorption isotherms (1). The forced flow velocity $\mathbf{u}$ is determined from Darcy's law

$$\mathbf{u} = -\mu \nabla P - \rho_a g \mathbf{k} \, . \tag{11}$$

The presure gradient ∇P is considered an average along the entire height of the sugar pile in the silo. Accordingly, we calculate an average convection flow $\mathbf{u}$. The constant g is the gravitational constant and $\mathbf{k}$ is the unit vector directed vertically upwards. The permeability is denoted by μ.

We need to specify initial conditions and proper boundary conditions for our problem. The boundary condition for the temperature is [4]

$$k\frac{\partial T}{\partial n} + H(T - T_{\text{out}}) = 0 \,, \tag{12}$$

which states that the heat flow across the silo walls is proportional to the temperature difference between the sugar at the silo surface and the outdoor temperature T_{out}. The proportionality constant H is the coefficient of surface heat transfer. The vector $\mathbf{n}$ is the outward unit normal at the surface of the sugar and $\partial/\partial n$ denotes an outward normal derivative. The surface heat-transfer constant will depend on the the silo wall material, degree of insulation, natural and forced convection. The boundary condition may also include a black-body radiation term, T^4-law. However, by linearization this effect can be absorbed in adjustments of H.

Boundary conditions for the moisture should also be given. Assuming no migration of moisture across the silo walls we have

$$\frac{D_v}{\tau}\frac{\partial}{\partial n}(\phi_v\rho_v) - \mathbf{n}\cdot\phi_v\rho_v\mathbf{u} = 0 \,. \tag{13}$$

Moreover, air does not flow across the silo walls and therefore $\mathbf{n}\cdot\mathbf{u} = 0$; Eqn.(13) reduces to

$$\frac{D_v}{\tau}\frac{\partial}{\partial n}(\phi_v\rho_v) = 0 \,.$$

In the case where conditioning air is blown into the sugar pile from below, the pressure p_a can be assumed fixed at the bottom and at the top of the silo of the stored sugar. From the ideal gas law we can then determine ρ_a at the bottom and at the top. Furthermore it is expected that the relative humidity of the air blown into the sugar is known and therefore p_v is also known from which ρ_v can be calculated. In the case where a considerable amount of moisture is transported through the silo walls, a suitable source term appears on the right-hand side of (13). At present we will not specify the boundary conditions in further detail as they will depend on a given silo geometry, construction materials, etc.

5 Scaling

The heat and the moisture migration equations can be written in dimensionless variables through the following scaling

$$\begin{aligned} x &= k_x \tilde{x}, \quad y = k_x \tilde{y}, \quad z = k_x \tilde{z}, \quad t = k_t \tilde{t}, \\ T(x,y,z,t) &= k_T \tilde{T}(\tilde{x},\tilde{y},\tilde{z},\tilde{t}), \quad W_s(x,y,z,t) = k_{ws} \tilde{W}_s(\tilde{x},\tilde{y},\tilde{z},\tilde{t}), \\ \rho_a(x,y,z,t) &= k_{\rho a} \tilde{\rho}_a(\tilde{x},\tilde{y},\tilde{z},\tilde{t}), \quad \rho_v(x,y,z,t) = k_{\rho v} \tilde{\rho}_v(\tilde{x},\tilde{y},\tilde{z},\tilde{t}), \\ p_a(x,y,z,t) &= k_{pa} \tilde{p}_a(\tilde{x},\tilde{y},\tilde{z},\tilde{t}), \quad p_v(x,y,z,t) = k_{pv} \tilde{p}_v(\tilde{x},\tilde{y},\tilde{z},\tilde{t}). \end{aligned} \tag{14}$$

The dimensionless variables are marked by a tilde and the constants have the same dimension as the original physical variables, that is k_x has dimension length, etc. In normalized variables the ideal gas law (2) for dry air can be written

$$\tilde{p}_a = \frac{k_{\rho a} R_a k_T}{k_{pa}} \tilde{\rho}_a \tilde{T} = \tilde{\rho}_a \tilde{T} \ , \tag{15}$$

by choosing the normalization constants in such a way that the factor in front of $\tilde{\rho}_a \tilde{T}$ equals unity. Out of the three independent normalization constants, we shall choose $k_{\rho a}$ and k_T according to the values in Table 1. The constant k_{pa} is then calculated from Eq.(15).

$k_{\rho a} = 1.225$ kg/m^3	$k_T = 298$ °K
$k_{pa} = 1.05 \cdot 10^5$ N/m$^2 \approx 1.$ atm.	$R_a = 287$ J/(kg °K)

Table 1.

The ideal gas law for the water vapour reads in normalized variables

$$\tilde{p}_v = \frac{k_{\rho v} R_v k_T}{k_{pv}} \tilde{\rho}_v \tilde{T} = \tilde{R}_v \tilde{\rho}_v \tilde{T} \ . \tag{16}$$

In this equation we have chosen $k_{\rho v} = k_{\rho a}$, $k_{pv} = k_{pa}$ and introduced the normalized gas constant $\tilde{R}_v$ for vapour given by

$$\tilde{R}_v = \frac{k_{\rho v} k_T}{k_{pv}} R_v \ . \tag{17}$$

Inserting the scaling (14) into the heat equation (6) we obtain

$$\frac{\partial \tilde{T}}{\partial \tilde{t}} + a_c \frac{\partial (\tilde{W}_s \tilde{T})}{\partial \tilde{t}} + a_1 \nabla \cdot (\tilde{\mathbf{u}} \tilde{p}_a) + a_2 \nabla \cdot (\tilde{\mathbf{u}} \tilde{p}_v) = \nabla^2 \tilde{T} + \frac{\partial \tilde{W}_s}{\partial \tilde{t}} \, . \tag{18}$$

The constants a_1, a_2 and the normalized air flow velocity $\tilde{\mathbf{u}}$ are given by

$$a_c = \frac{C_w}{C_s} k_{ws}, \quad a_1 = \varepsilon \frac{C_a}{C_s} \frac{k_{\rho a}}{\phi_s \rho_s}, \quad a_2 = \varepsilon \frac{C_v}{C_s} \frac{k_{\rho v}}{\phi_s \rho_s} \frac{1}{\tilde{R}_v}, \quad \tilde{\mathbf{u}} = \frac{k_t}{k_x} \mathbf{u}, \tag{19}$$

and

$$k_x^2 = \frac{k}{\phi_s \rho_s C_s} k_t, \qquad k_{ws} = \frac{C_s}{l} k_T. \tag{20}$$

All normalization constants are thus fixed except the time scaling k_t. The processes we are interested in occur over a time scale of order months. Therefore we choose k_t to be 30 days ($k_t = 2.59 \cdot 10^6$s). Typical values for the physical constants entering our problem are stated in Table 2. Some of them are given by the values found in [1]. As an example the diffusivity of water vapour D_v is calculated from [3]

$$D_v = \frac{9.1 \cdot 10^{-9} T^{2.5}}{T + 245.18} \, . \tag{21}$$

$k = 0.3$ J/(s m °K)	$\phi_s = 0.5$
$\rho_s = 1.610^3$ kg/m^3	$C_s = 1.2$ kJ/(kg °K)
$R_v = 462$ J/(kg °K)	$C_a = 1.01$ kJ/(kg °K)
$C_v = 1.88$ kJ/(kg °K)	$C_w = 4.18$ kJ/(kg °K)
$\varepsilon = 0.4$	$l = 2440$ kJ/kg
$D_v = 2.57 \cdot 10^{-5} \text{m}^2/\text{s}$	$\tau = 1.53$

Table 2.

The moisture migration equation (9) written in terms of the normalized variables becomes

$$\frac{\partial \tilde{W}_s}{\partial \tilde{t}} + b_1 \nabla \cdot \left(\tilde{\mathbf{u}} \frac{\tilde{p}_v}{\tilde{T}} \right) = b_2 \nabla^2 \left(\frac{\tilde{p}_v}{\tilde{T}} \right) , \tag{22}$$

and here we have introduced the constants

$$b_1 = \frac{\varepsilon k_{pv}}{\phi_s \rho_s k_{ws} R_v k_T} \quad \text{and} \quad b_2 = \frac{\varepsilon k_t D_v k_{pv}}{\phi_s \rho_s k_{ws} k_x^2 \tau R_v k_T} . \tag{23}$$

In normalized coordinates the continuity equation for air reads

$$\frac{\partial \tilde{\rho}_a}{\partial \tilde{t}} + \nabla \cdot (\tilde{\rho}_a \tilde{\mathbf{u}}) = 0 , \tag{24}$$

and the sorption isotherms are described by

$$\tilde{p}_v = \tilde{p}_v(\tilde{W}_s, \tilde{T}) . \tag{25}$$

With k_t equal 30 days the spatial scaling factor k_x takes the value 0.901 m using the physical variables stated in Table 1 and 2. As the size of a silo is large compared with k_x, this means that spatial dependencies of the temperature, humidity, etc. are predominant at the boundaries of the silo and accordingly we expect that the model can be treated as a boundary layer problem.

The distance scaling indicated by (20) comes from balancing heat conduction with rate of change of the temperature. It also gives a similar balance in the water equation, e.g. (22). Over the lengths comparable with larger silos (say height or radius of the order of 10 m), these terms cannot balance – except in boundary layers, for instance near the silo wall. However, it is found, for speeds of flow used in forced convection, that the convective term (in the upward, z, direction) still balances the time rate of change in (22). Thus, returning to dimensional variables, away from the boundary layer(s) temperature T is constant (to leading order) while water content satisfies, approximately,

$$\phi_s \rho_s \frac{\partial W_s}{\partial t} + \varepsilon u \frac{\partial}{\partial z} \left(\frac{p_v}{R_v T} \right) = 0,$$

with p_v and W_s being related through the relevant sorption isotherm; this is a nonlinear, first-order, hyperbolic equation. (With ε and μ being constant, $\mathbf{u}$ is constant and directed vertically upwards, the z direction.) In the boundary layer near the silo wall the diffusive terms in the horizontal, say x, direction (normal to the wall) now play a role and the two-dimensional problem,

$$\phi_s \rho_s C_s \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} , \quad \phi_s \rho_s \frac{\partial W_s}{\partial t} + \varepsilon u \frac{\partial}{\partial z} \left(\frac{p_v}{R_v T} \right) = \frac{\varepsilon D_v}{\tau} \frac{\partial^2}{\partial x^2} \left(\frac{p_v}{R_v T} \right) ,$$

has to be solved. (The space variables x and z should be scaled differently from each other in the nondimensional problem.) It is seen that T is independent of z (away from any boundary layers near the top and bottom of the sugar) and, as it satisfies the linear heat equation, can be determined explicitly. The remaining problem for the moisture is then a nonlinear convection diffusion equation (the two transport mechanisms take place in different directions). It is also inhomogeneous due to the presence of $T(x, t)$.

It should be noted that the other terms, including the effects of latent heat and of specific heat of adsorbed moisture, were found, on carrying out the scaling, to be negligible.

In solving the coupled equations (18, 22, 24, 25, 15) more generally, for a given silo geometry, a numerical approach seems to be the only adequate way to arrive at a useful and practical solution. A simple and also reliable method is the method of lines. Here the space is discretized according to $x = i\,dx$, $i = 0, 1, ..., n_x$, and similarly for the y- and z-directions. All the dependent variables are defined at discrete points (i, j, k). Spatial derivatives are replaced by central differences of second order in the discretization lengths dx, dy and dz, and inserted into the governing system of partial differential equations. The resulting system of coupled ordinary differential equations (ODEs) can now be solved numerically by using standard ode solvers. Alternative methods are e.g. the finite element method which is particularly suitable for silos with complex geometry. In reference [1] a control volume discretization procedure has been employed in order to solve moisture migration in stored grain. This approach could be adopted to our problem and formulated in such a way that silos with complex geometry can be handled.

Concerning equation (25) we need an analytical function of the sugar moisture content and temperature which approximates the sorption isotherms shown in Fig.1. For grain an example is presented in [1].

6 Concluding Remarks

The main result of the above work is the derivation of a mathematical model for moisture migration through stored sugar. The model includes air conditioning of the sugar. We have assumed thermodynamic equilibrium between the water in the sugar and the vapour in the interstitial air described by the sorption isotherms. This assumption may be invalid if air conditioning is employed with a high flow velocity, due to the time necessary to equilibrate. The flow of the conditioning air is determined from Darcy's law. This equation can be solved in conjunction with the heat and moisture equations. However, it is believed that the air flow is approximately constant, and accordingly Darcy's law is used to calculate a uniform flow velocity from the bottom to the top of the silo. Scaling of the governing equations indicates that the problem has at least one boundary layer.

The next step is to solve the model equations. For practical and realistic situations we should apply numerical methods in computer simulations. A number of methods are available. We have mentioned three of which the method of lines may be the fastest and most easy to implement on a computer. This method is suitable for silo geometries which are not too complex. Otherwise one could use the finite element method or a finite volume method for more complex geometries. The content of sugar in a silo is lovered through the year and the top surface of the sugar pile may be rather irregular.

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The Group Working on the Problem

Janis Cepitis, [cepitis@lanet.lv]

Erik Bent Hansen, [E.B.Hansen@mat.dtu.dk]

Andrew Lacey, [A.A.Lacey@ma.hw.ac.uk]

Tiina Roose, [roose@maths.ox.ac.uk]

Mads Peter Sørensen, [mps@imm.dtu.dk]

Jannik Vindeløv, [javi@kvl.dk]