Bond Sweeteners

Riskcare

1 Background

A company that is interested in financial expansion may acquire additional capital by issuing new company shares or bonds. To sweeten the bonds, often the company attaches embedded options that allow the bondholder to exchange the bond for shares. If, upon exercise of this embedded option, new shares are created, the option is called a warrant. Exercise of a warrant will therefore increase the total number of shares in the market. Note that the asset value per share may be less after exercise than before. We shall consider American-style warrants which may be exercised at any time.

Consider the holder of such a warrant and suppose that upon exercise there is a dilution effect (the share price falls). Such a person could short sell shares immediately prior to exercising, then buy them back immediately after, thereby realising a risk-free profit (assuming the exercise is in some sense optimal). This represents an arbitrage opportunity and therefore cannot occur in a perfect market. (If it is "sub-optimal" to exercise, we shall show tht the share price moves against the exerciser, so eliminating the arbitrage possibility.)

In order to understand this, we require a simple model for the total equity capital (assets − liabilities).

We model the asset value, A, which is the total net assets excluding warrants, as a log-normally distributed random variable satisfying

$$\frac{dA}{A} = \mu dt + \sigma dX. \tag{1}$$

The total equity capital is $A - N_w W$ where N_w is the number of outstanding warrants and W is the price of a single warrant. This must be equal to the total value of all shares in the market, $N_s S$, where N_s is the number of shares in the market and S is the price of a single share:

$$N_s S = A - N_w W. \tag{2}$$

Now, consider what happens if n warrants are exercised. The warrant liabilities fall to $(N_w - n)W$ and A increases to $A^- + nK$ where K is the strike price of the warrant and A^- is the value of A immediately prior to exercise. The number of shares jumps to $N_s + n$. At all times, to avoid arbitrage, we must have $W - S + K \geq 0$ (otherwise it would be possible to buy and exercise the warrant for less than the price of a share).

Immediately before and after the exercise of n warrants we have

$$\begin{aligned} N_s S^- &= A^- - N_w W && \text{(immediately before)},\\ (N_s + n)S^+ &= A^- + nK - (N_w - n)W && \text{(immediately after)}. \end{aligned}$$

As implied above, S^+ must be greater than or equal to S^- or there is an arbitrage opportunity. This gives

$$W - S + K \leq 0.$$

Hence, to avoid both arbitrages, the warrant should be exercised only if

$$W - S + K = 0.$$

If the warrant holder exercises prematurely, ie $W > S - K$, $S^+ > S^-$ and the existing shareholders benefit at the exerciser's expense (unless the warrant holder holds ALL the shares, the generous donation of $W - S + K > 0$ to the company's funds cannot be recouped by selling shares).

Assuming the random walk given in equation (1), the classical Black-Scholes analysis can be applied. Thus, with a portfolio

$$\Pi = W(A,t) - \Delta S(A,t), \tag{3}$$

where Δ is the hedge ratio, we can construct an instantaneously risk-free portfolio as follows. We find

$$d\Pi = dW - \Delta dS - \Delta D dt + C dt, \tag{4}$$

where D is the dividend paid to shareholders and C is the coupon paid to warrant holders. Using equation (2) to eliminate dS, Itô's lemma to expand dW, and choosing Δ to eliminate the random component in $d\Pi$ (that corresponding to dX), we find that

$$d\Pi = \left(\frac{1}{1 - N_W W_A}\left(W_t + \frac{1}{2}\sigma^2 A^2 W_{AA}\right) - \frac{N_S W_A}{1 - N_W W_A} D + C\right) dt, \tag{5}$$

with the choice

$$\Delta = \frac{N_S W_A}{1 - N_W W_A}. \tag{6}$$

(Δ represents the theoretical dynamic hedging strategy for a holder of a warrant.) Since the warrant may be exercised early, it *may* be optimal, under certain circumstances, to exercise early, in which case an arbitrageur would have the warrant exercised against them to their detriment. Thus there are, potentially, situations where the rate of return on the risk-free Δ hedged portfolio is inferior to the risk-free bank rate and arbitrage pressure to realign the rates is not possible.

Clearly, on the other hand, the return cannot exceed the return from the bank (or the pressure of borrowing would force up the risk-free bank rate or force down the return on Π) and hence the best we can do is

$$d\Pi \leq r\Pi dt.$$

In region of (A,T)-space where $d\Pi = r\Pi dt$, it is best to hold the warrant and in regions where $d\Pi < r\Pi dt$ it is optimal to exercise the warrant for the following reasons. If $W > S - K$, the warrant would not be exercised (for a profit of $S - K$), but rather sold (for W) or held. Hence

$$d\Pi = r\Pi dt$$

in that case. Conversely if $d\Pi < r\Pi dt$, then W cannot be bigger than $S - K$ (otherwise you would sell). Equally, W cannot be less than $S - K$ (this provides arbitrage). Hence

$$W = S - K. \tag{7}$$

Thus we have the linear complementarity problem

$$\begin{aligned} d\Pi &\leq r\Pi dt \\ W &\geq \max(S - K, 0) \\ (d\Pi - r\Pi dt)(W - \max(S-K,0)) &= 0. \end{aligned} \tag{8}$$

W must be continuous in order to avoid arbitrage and Δ must be continuous for the same reason. This is a classical LCP which is equivalent to a variational inequality well suited to numerical analysis (see Elliott & Ockendon (1982) or Wilmott *et al.* (1993)).

Substituting for Π and S, we arrive at

$$W_t + \tfrac{1}{2}\sigma^2 A^2 W_{AA} + (rA - DN_S)W_A - rW + C(1 - N_W W_A) \leq 0, \tag{9}$$

$$W \geq \max\left(\frac{A - N_S K}{N_S + N_W}, 0\right). \tag{10}$$

An important consequence of this equation is that the volatility to consider in its solution is associated with A rather than the volatility of the share price. If N_W is large, the asset volatility can be substantially larger than the stock volatility.

A realistic model for the dividend is that it is proportional to the share price

$$D = \overline{D}S = \frac{\overline{D}}{N_S}(A - N_W W),$$

which makes equation (9) nonlinear. Nevertheless, its numerical solution should not present any difficulties.

In the rest of this report we attempt to determine whether it is optimal to exercise all the warrants at the first point at which $W = S - K$ or only some of the warrants. In the first instance we consider a "perfect market" world as considered above and in the second we allow a situation in which "dumping" of newly created shares (as a result of exercise) causes a fall in the price of the shares (as a result of supply and demand in the share market) – an illiquid market model.

2 Exercise strategy

Consider now the case where we hold $n(t)$ warrants; $n(t)$ represents our exercise strategy and $\dot{n}(t) \leq 0$. (Typically $n(t)$ can be represented as

$$n(t) = n_0 - \int_0^t \phi(S(\tau), \tau)\, d\tau,$$

where $\phi(S(\tau), \tau) \geq 0$.) We form a Δ hedged portfolio Π with value

$$\Pi = nW - \Delta S$$

and we find that

$$d\Pi = Wdn + ndW + Kdn - Sdn - \Delta dS + nCdt - \Delta Ddt$$

where, as before, C represents the coupon payment from the warrant and D the dividend on the share. The terms Kdn and $-Sdn$ arise from the cost of exercising the warrant and the change in value of S as a result of the exercise. Similarly, we find that

$$dS = \frac{dA}{N_S} - \frac{N_W}{N_S}dW + \frac{1}{N_S}(S - K - W)dn$$

where the $S-K-W$ term arises from the change in value in S resulting from the exercise of dn warrants. After applying Itô's lemma to expand dW we find

$$\begin{aligned} d\Pi &= \frac{1-(N_W-n)W_A}{1-N_W W_A}(W+K-S)dn \\ &+ \frac{n}{1-N_W W_A}(W_t + \frac{1}{2}\sigma^2 A^2 W_{AA} + (1-N_W W_A)C - N_S W_A D)dt \\ &\leq r\Pi dt; \end{aligned} \qquad (11)$$

the inequality, as before, arising because it may be optimal to exercise the warrant early. The elimination of arbitrage shows that

$$W \geq S - K$$

and, as before, we arrive at a linear complementary problem.

If we consider (11) as a partial differential inequality we find that

$$\begin{aligned} W_t + \frac{1}{2}\sigma^2 A^2 W_{AA} + (rA - N_S D - N_W C)W_A - rW + C \\ \leq (1-(N_W-n)W_A)(S-K-W)\frac{\dot{n}}{n} \end{aligned} \qquad (12)$$

Since $S - K - W \leq 0$, $\dot{n}/n \leq 0$ and $(1-(N_W-n)W_A) \geq 0$ we can maximize the value of $W(A,t)$, for time $t < T$ (expiry) by choosing $(S-K-W)\dot{n} = 0$; that is, by only exercising ($\dot{n} \neq 0$) when $(S-K-W) = 0$. This is easiest to see by considering the situation in which $W > S - K$, in which case we have equality in (12), and recalling that (12) is solved backwards in time from a given payoff at expiry. Unfortunately, this analysis does *not* indicate whether or not it is optimal to exercise ALL the warrants when $W = S - K$ or only some of them.

3 Illiquid market model

In an illiquid market it is not possible to sell large parcels of stocks immediately at the quoted price. Instead, sales depress the price and a large parcel must be sold in bits, to realise less than the initial price. Also, bid-offer spreads are usually large.

Suppose that warrant-holders who exercise immediately sell the stock, perhaps because they do not want to hold it long-term. Then there *is* a drop in share price on exercise of warrants. (Note that this effect is *not* classical dilution in the sense that it results from *selling* the shares and not from the increase in the total number of shares.)

(Note (i) that the *small* transactions involved in a day-to-day hedging strategy do not cause drops; (ii) that the writer (the company) does not hedge the warrants; (iii) existing stock holders cannot sell large amounts ahead of an anticipated exercise (calculated by

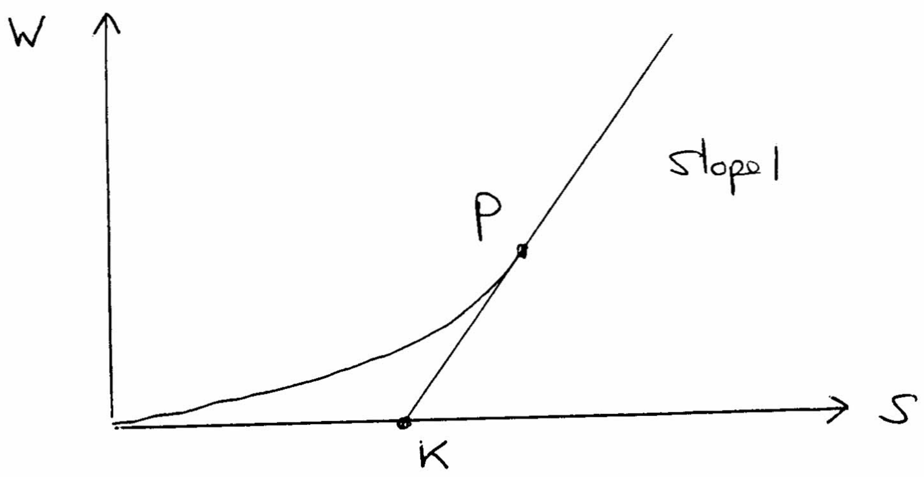

Figure 1:

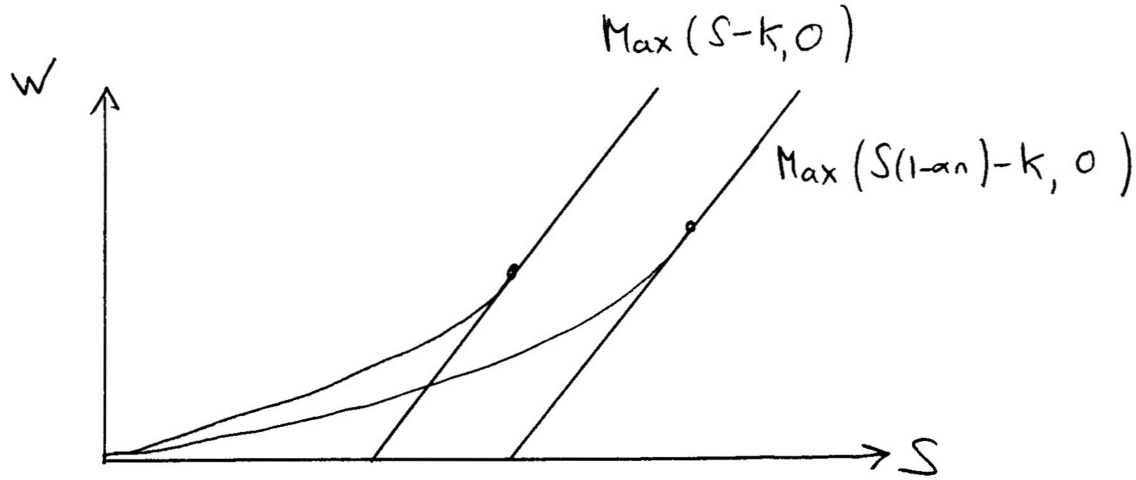

Figure 2:

solving the problem below) with consequent drop in the stock price, because the sales will trigger a drop that will forestall exercise – and so defeat their object.)

Let us model the warrants as American call options on dividend-paying stock (to avoid complications as above). Without the drop, the payoff is $\max(S-K,0)$ and the value $W(S,K)$ looks like figure 1, where P indicates the optimal exercise price.

Suppose the exercise of a proportion n of the warrants, followed by stock dumping, causes a drop $\alpha n S$ in S, where α is a small (?) constant ($0 \leq \alpha n \ll 1$). The effective payoff is $\max(S(1-\alpha n)-K,0)$; see figure 2. This looks like $(1-\alpha n)$ call options with exercise price $K/(1-\alpha n)$ and can be computed.

Is it best to exercise all the warrants at once ($n=1$)? Since exercise decreases S, we may move out of the optimal exercise regime. Try a strategy of: exercise half of the warrants at the first opportunity (if it occurs) and the other half at the next (if it occurs). Does this give a higher warrant value?

Working back from expiry and assuming at least two opportunities for early exercise will occur, when $\frac{1}{2}$ of the warrants are still alive the value of W is

$$(1-\frac{1}{2}\alpha n)V_{AM}(S,t;K/(1-\frac{1}{2}\alpha n))$$

where V_{AM} is the value of an American call. This is nearly the "payoff" for the warrants while they are all alive – ie what you exercise into. But we must remember that S jumps to $S(1-\frac{1}{2}\alpha n)$ when the first tranche of warrants is exercised. Thus we have a payoff of $(1-\frac{1}{2}\alpha n)V_{AM}((1-\frac{1}{2}\alpha n)S,t;K/(1-\frac{1}{2}\alpha n))$, ie $(1-\frac{1}{2}\alpha n)^2V_{AM}(S,t;K/(1-\frac{1}{2}\alpha n)^2)$ (I think). The warrant value should be greater than this payoff, which should be compared with $(1-\alpha n)V_{AM}(S,t;K/(1-\alpha n))$, which is the warrant value when the strategy is to exercise them all at once. Since $(1-\frac{1}{2}\alpha n)^2 > 1-\alpha n$, for the two-exercise strategy, there are more of the equivalent Black-Scholes call values and they have a lower strike price, than for one-exercise. So, exercising twice appears to be better. What remains is to determine the exercise strategy, as a function of S, W and t that does best of all.

4 References

Elliott, C M & Ockendon, J R. Weak and variational methods for free and moving boundary problems, 1982, Pitman.
Wilmott, P, Dewynne, J N, Howison, S D. Option pricing: mathematical models and computation, 1993, Oxford Financial Press.

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