

Friction welding

Rolls Royce

1 The problem

A novel welding process is investigated in which two work-pieces are vibrated in a sliding motion while being pressed together. The sliding generates heat which softens a thin layer of material, which can then deform in plastic flow. Applying the pressure produces a squeeze-film flow, which removes impurities on the original surface, and which also plays a controlling role by trapping heat in the thin layer, thereby determining its thickness.

2 Simplifications

In this first study we make a number of simplifications. First, we examine a two-dimensional problem, appropriate for work-pieces long in the third direction. Second, we will only study the steady state. Third, we assume that the displacement of the sliding motion is small compared to the width, so that we can ignore end-effects and ignore any variations in the applied pressure while the load is held constant. Fourth, we take the variation in time of the sliding motion to be a square wave. Of course the real sliding is more nearly sinusoidal, but the constant magnitude of the shear-rate in a square wave allows us to ignore fluctuations in the heat generated. Fifth, we assume the the squeezing velocities are smaller than the sliding velocities. This means that heat is generated only by the sliding motion. Further, the plastic flow which has a nonlinear rheology can be linearised around the base sliding motion, thus yielding a simpler decoupled problem for the small squeeze flow. Sixth, we ignore inertia in the flow in the thin softened layer.

3 Rheology

The plastic flow in the softened layer is predominantly a shearing motion. We assume that the shear rate $\dot{\gamma}$ is related to the shear stress σ by a power

law

$$\sigma = \kappa(T)\dot{\gamma}^\alpha. \quad (1)$$

The power law index α is positive and less than unity. For metals it is typically $\frac{1}{4}$. The temperature dependency of the material modulus $\kappa(T)$ has a rapidly varying exponential factor corresponding to a high activation temperature T_a and a more slowly varying linear factor which vanishes at a melting temperature T_m . It is convenient for the asymptotic analysis later to refer the two factors to the lower melting temperature. Thus we take

$$\kappa(T) = \kappa_m \left(1 - \frac{T}{T_m}\right) \exp\left(\frac{T_a}{T_m} \left(\frac{T_m}{T} - 1\right)\right). \quad (2)$$

All temperatures here are measured relative to absolute zero.

4 Governing equations

Let x be measured in the sliding direction, and y across the softened layer, with the origin at the centre. We take the width of the work-piece to be $2L$, so that the ends of the flow region are $x = \pm L$.

The temperature $T(x, y, t)$ is governed by the heat equation with advection, diffusion across the thin layer, and heat generation. Thus with density ρ , specific heat c_p and thermal conductivity k

$$\rho c_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}\right) = k \frac{\partial^2 T}{\partial y^2} + \sigma \dot{\gamma}. \quad (3)$$

Outside the thin heated layer, the temperature assumes the ambient value

$$T \rightarrow T_0 \quad \text{as} \quad y \rightarrow \pm\infty. \quad (4)$$

The plastic flow (u, v) in the thin softened layer, a combination of the sliding and squeezing motions, we take to be governed by the standard lubrication approximation in which inertia is ignored. Thus the flow is predominantly a shearing flow, with shear rate $\dot{\gamma} = \partial u / \partial y$. This shearing produces a shear stress σ given by equation (1). Any variations of the shear stress across the layer are generated by a pressure gradient along the layer

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \sigma}{\partial y}. \quad (5)$$

Mass conservation gives the smaller flow v across the layer

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad (6)$$

The boundary conditions on the flow outside the softened layer are

$$u \rightarrow \pm U_0 \text{sign}(\cos \Omega t), \quad v \rightarrow \mp V \quad \text{as} \quad y \rightarrow \pm \infty, \quad (7)$$

where $2U_0$ is the imposed sliding velocity of one work-piece over the other, Ω the frequency, made into a square wave by the application of the function $\text{sign}(\cdot)$, and V is the constant velocity at which the work-pieces approach, caused by the applied pressure. It is our goal to predict this approach velocity. At the ends of the layer, where the squeeze flow escapes, the pressure is set to atmospheric

$$p = 0, \quad \text{at} \quad x = \pm L. \quad (8)$$

It is convenient to express the externally imposed squeezing load as an average pressure P ,

$$\frac{1}{2L} \int_{-L}^L p \, dx = P. \quad (9)$$

The problem is to predict the squeezing velocity V , the thickness of the softened layer, its temperature and the difference of temperature across it. The given parameters are the sliding motion U_0 and Ω , the squeezing pressure P , the size of the work-piece L , the ambient temperature T_0 , and the material properties ρ , c_p , k , $\kappa(T)$ and α . As a concrete example consider: $U_0 = 0.22 \text{ m s}^{-1}$, $\Omega = 188 \text{ s}^{-1}$, $P = 1.57 \times 10^8 \text{ Pa}$, $L = 6 \times 10^{-3} \text{ m}$, $\rho = 4420 \text{ kg m}^{-3}$, $c_p = 750 \text{ m}^2 \text{ s}^{-2} \text{ }^\circ\text{K}^{-1}$, $k = 15 \text{ kg m s}^{-3} \text{ }^\circ\text{K}^{-1}$, $\alpha = 0.25$, $\kappa_m = 5.4 \times 10^8 \text{ Pa s}^{0.25}$, $T_m = 1350 \text{ }^\circ\text{K}$, and $T_a = 8400 \text{ }^\circ\text{K}$.

5 Non-dimensionalisation

This is a first attempt to find the scalings of the unknowns. Refinements will emerge later in the asymptotic analysis. The non-dimensionalisation has been designed not to depend heavily on the particular rheology.

It is clear that distance along the layer x should be scaled with the half-width L , the velocities along the layer u scaled with the imposed sliding U_0 , and so that time t is scaled with L/U_0 . Further the pressure p should be scaled with the imposed average pressure P . We chose to scale temperature T with the melting temperature T_m .

The scaling for the thickness H of the softened layer is not obvious. If we suppose temporarily that we knew H , then we would scale the shear-rate $\dot{\gamma}$ with U_0/H , the velocities v across the layer with U_0H/L , and the shear stresses σ with PH/L . With these estimates we can balance heat diffusion and heat generation

$$\frac{kT_m}{H^2} = \frac{PHU_0}{LH}.$$

This gives an estimate for the thickness of the softened layer

$$H = \sqrt{\frac{kT_m L}{PU_0}}. \quad (10)$$

As we shall see later, this scaling is false because the variation of the temperature across the softened layer is much smaller than T_m .

The above scalings produce seven non-dimensional groups. First, a non-dimensional frequency

$$\omega = \Omega L/U_0$$

which is $O(1)$ and does not enter the subsequent analysis. Second, the aspect ratio of the layer

$$H/L = \sqrt{kT_m/LPU_0}$$

which we have assumed to be small, and as such also does not enter the subsequent analysis. Third and fourth are two ratios of temperature

$$\mathcal{T}_a = T_a/T_m \quad \text{and} \quad \mathcal{T}_0 = T_0/T_m,$$

which we shall treat as $O(1)$ parameters. Fifth, a Péclet number

$$Pe = \rho c_p V H/k = \rho c_p T_m/P.$$

This is a usefully large $O(1)$ parameter. Sixth, the power law index α . Seventh, a parameter which measures the strength of the material compared with the applied forces. From the scaling of the shear stress PH/L and the scale of the shear-rate U_0/H , we can construct a scaling for the modulus of the plastic flow $PH/L(U_0/H)^\alpha$. Comparing this measure to the modulus of the material κ_m , we obtain the non-dimensional group

$$\mathcal{K} = \frac{\kappa_m L U_0^\alpha}{P H^{1+\alpha}} = \frac{\kappa_m (L/P)^{(1-\alpha)/2} U_0^{(1+3\alpha)/2}}{(kT_m)^{(1+\alpha)/2}}.$$

For the example data, $\omega = 5.1$, $H/L = 0.31$, $\mathcal{T}_a = 6.2$, $\mathcal{T}_0 = 0.22$, $Pe = 28$, $\alpha = \frac{1}{4}$ and $\mathcal{K} = 36$.

Henceforth we shall work with non-dimensional variables, for which we will use the same symbols. The governing equations (1 – 9) are little changed by this non-dimensionalisation: κ_m , T_m and T_a/T_m in (2) are replaced by \mathcal{K} , 1 and \mathcal{T}_a , ρc_p and k in (3) are replaced by Pe and 1, U_0 and Ω in (7) are replaced by 1 and ω , and L and P in (8) and (9) are replaced by 1.

In a later section we shall make an asymptotic theory for strong materials $\mathcal{K} \gg 1$.

6 Sliding dominating squeezing

In this section we present the scheme for solving the equations when the sliding is larger than squeezing. The difficulty to be overcome is the nonlinear rheology, which couples the two motions. Assuming that one dominates enables us to decouple the two, and to obtain for the smaller a problem linearised around the larger.

The dominant sliding motion is driven by an externally applied shear stress, say $S \text{sign}(\cos \omega t)$. A simple solution can be constructed with the sliding velocity and the temperature fields both independent of position x along the softened layer. With temperature only a function of position y across the layer, the modulus $\kappa(T)$ only varies across the layer. Hence the shear-rate $\dot{\gamma}$ only varies across the layer. And finally we conclude that the heat generated also only varies across the layer, which is consistent with the original assumption that the temperature varies only across the layer. With the sliding motion independent of position along the layer, there is no need for any pressure gradient to keep the net flux constant. Hence the shear stress is constant across the layer.

The problem for the sliding motion $\tilde{u}(y) \text{sign}(\cos \omega t)$ therefore reduces to

$$\frac{\partial \tilde{u}}{\partial y} = S^{1/\alpha} \kappa^{-1/\alpha}(T(y)). \quad (11)$$

The unknown shear stress S is fixed by the boundary conditions from (7)

$$\tilde{u} \rightarrow \pm 1 \quad \text{as} \quad y \rightarrow \pm \infty.$$

By symmetry $\tilde{u}(0) = 0$.

We consider the squeeze flow to produce small changes in the shear-rate, from $\dot{\gamma}$ to $\dot{\gamma} + \delta\dot{\gamma}$. Linearising the rheology for the small changes, the change in shear-rate produces a change in shear stress, from σ to $\sigma + \delta\sigma$, given by

$$\delta\sigma = \mu \delta\dot{\gamma}, \quad \text{with} \quad \mu = \alpha \kappa(T) \dot{\gamma}^{\alpha-1}.$$

We evaluate this effective viscosity μ using the dominant sliding motion for the shear-rate, *i.e.*, $\dot{\gamma} = \partial\bar{u}/\partial y$. Thus using (11)

$$\mu = \alpha S^{(\alpha-1)/\alpha} \kappa^{1/\alpha} \quad (12)$$

The squeezing flow is therefore governed by a linear lubrication problem with a viscosity μ which only varies across the softened layer.

Now the squeezing motion brings the work-pieces together at the approach velocity $\pm V$, which is independent of position along the softened layer because the work-pieces are solid. This will produce a flux of softened material along the layer proportional to distance x from the centre. We therefore seek a simple solution to the squeeze flow which varies across the layer and is proportional to distance along the layer, $x\bar{u}(y)$. This is possible, because the effective viscosity μ is independent of position along the layer. To drive the flux velocity proportional to x a pressure gradient is required, also proportional to x . Thus the pressure varies quadratically along the layer, vanishing at the ends by (8). Given the load (9) and the non-dimensionalisation, the pressure must be

$$p = \frac{3}{2}(1 - x^2).$$

The squeeze flow $x\bar{u}(y)$ is governed by

$$0 = -3 + \frac{\partial}{\partial y} \left(\mu \frac{\partial \bar{u}}{\partial y} \right).$$

Integrating this from $y = 0$ where $\partial\bar{u}/\partial y$ vanishes by the symmetry of the squeeze flow, and then substituting the expression (12) for the effective viscosity, we obtain

$$\frac{\partial \bar{u}}{\partial y} = -3y\alpha^{-1} S^{(1-\alpha)/\alpha} \kappa^{-1/\alpha}. \quad (13)$$

This must be solved subject to boundary condition from (7)

$$\bar{u} \rightarrow 0 \quad \text{as} \quad y \rightarrow \pm\infty.$$

The unknown value $\bar{u}(0)$ is selected so as to satisfy this condition.

From the squeeze velocity along the layer, $x\bar{u}(y)$ and the mass conservation (6), we obtain a equation for squeeze velocity across the layer

$$\frac{\partial v}{\partial y} = -\bar{u}. \quad (14)$$

This is to be integrate from the centre line where by symmetry of the squeeze flow

$$v = 0 \quad \text{at} \quad y = 0,$$

in order to find the velocity V at which the work-pieces approach,

$$v \rightarrow \mp V \quad \text{as} \quad y \rightarrow \pm\infty.$$

Finally we can return to the non-dimensionalised version of heat equation (3), now with expressions for the heat generated by the sliding motion and with advection from the squeezing motion.

$$Pe v \frac{\partial T}{\partial y} = \frac{\partial^2 T}{\partial y^2} + S^{(1+\alpha)/\alpha} \kappa^{-1/\alpha}. \quad (15)$$

This has to be solved subject to the boundary condition

$$T \rightarrow \mathcal{T}_0 \quad \text{as} \quad y \rightarrow \pm\infty.$$

When the sliding dominates the squeezing, we needs to solve equations (11), (13), (14) and (15), a coupled system of ordinary differential equations, to obtain the profiles of the sliding and squeezing motion, \tilde{u} and \bar{u} , the velocity of approach to the work-pieces V , and the temperature profile T . The condition that the sliding dominates the squeezing is that $\bar{u} \leq O(\tilde{u}) = 1$.

7 Scaling for hard materials, $\mathcal{K} \gg 1$

When the material is hard, in the sense that its modulus of the plastic flow κ_m is much greater than the scaling modulus $PH/L(U_0/H)^\alpha$; the material becomes sufficiently soft to flow only very close to the melting temperature. At such temperatures, the variation in the modulus comes from the linear factor in (2) while the exponential factor varies little. Thus in the softened layer we can approximate the non-dimensional version of (2) by

$$\kappa(T) \sim \mathcal{K}(1 - T). \quad (16)$$

We shall find that the difference between the temperature on the centreline and the melting temperature

$$\Delta T = 1 - T(0)$$

is much smaller than $1/\mathcal{K}$, so that κ is also small.

With the linear variation (16) of the modulus, its value will be double the centreline value $\mathcal{K}\Delta T$ where the temperature is twice as far below the melting temperature

$$T = 1 - 2\Delta T.$$

At this point, the factor $\kappa^{-1/\alpha}$ in the equations (11), (13) and (15) has decreased by a factor of 16 when $\alpha = \frac{1}{4}$, and so beyond this point the sliding, squeezing and heating become negligible compared with that on the centreline. The temperature variation across the softened layer is thus merely ΔT . We shall find the contrast between this small temperature difference and the large difference between ambient and melting explains why the softened layer is very thin for hard materials.

We now seek the new scalings of the variables of interest in the limit of hard materials $\mathcal{K} \gg 1$, working in terms of the non-dimensional problem in §5. Let h be the unknown (non-dimensional) thickness of the softened layer. The shear-rate in the sliding motion (11) is then $\partial\tilde{u}/\partial y = O(1/h)$. Thus the effective viscosity (12) is $\mu = O(\alpha\kappa h^{1-\alpha})$. The squeeze flow velocity $\bar{u} = O(h^2/\mu)$ produces an approach velocity $V = O(h^3/\mu) = O(\alpha^{-1}\kappa^{-1}h^{2+\alpha})$. We substitute these estimates into the heat equation (15), being careful to distinguish between the temperature difference $1 - \mathcal{T}_0$ of the material advected towards the softened layer and the temperature difference ΔT which diffuses across the layer. Thus

$$Pe(1 - \mathcal{T}_0)\alpha^{-1}\kappa^{-1}h^{1+\alpha} = Pe\frac{V}{h}(1 - \mathcal{T}_0) = \frac{\Delta T}{h^2} = \sigma\dot{\gamma} = \kappa h^{-(1+\alpha)}.$$

Finally recalling the temperature dependence of the modulus (16), we have

$$\Delta T = \kappa/\mathcal{K}.$$

Solving for the unknowns, we find the scalings in the softened layer

$$\kappa = \beta\mathcal{K}^{-(1+\alpha)/(1-\alpha)}, \quad \Delta T = \beta\mathcal{K}^{-2/(1-\alpha)}$$

$$h = \mathcal{K}^{-1/(1-\alpha)}, \quad V = (\alpha\beta)^{-1}\mathcal{K}^{-1/(1-\alpha)}, \quad \text{and} \quad \bar{u} = (\alpha\beta)^{-1}, \quad (17)$$

where $\beta^2 = Pe(1 - \mathcal{T}_0)/\alpha$. We will return in §9 to discuss the consequences of these scalings on the original dimensional variables of interest. We comment here that harder materials have thinner layers and slower approach velocities.

We now see that for hard materials ($\mathcal{K} \gg 1$) that the modulus in the softened layer is indeed small ($\kappa \ll 1$) because the temperature on the centreline is close to melting ($\Delta T \ll 1$). The thickness of the softened layer

is also much smaller than our earlier estimate H in (10) by the new factor $h \ll 1$. The squeezing flow is however $O(1)$, as $\mathcal{K} \rightarrow \infty$. However there is a useful factor $(Pe(1 - \mathcal{T}_0)\alpha)^{1/2}$ to keep it smaller than the sliding motion.

In making the above estimates of the terms in the heat equation, we used the small temperature differences ΔT in the diffusion across the layer and the large temperature difference $1 - \mathcal{T}_0$ in the advection towards the layer. The latter is necessary so that the rate of heat generation is equal to the specific heat required to raise the temperature from ambient to near melting. The former gives the temperature differences necessary to diffuse heat out of the softened layer. A consequence of using these two scales is that within the softened layer, where one should use the smaller scale, the advection term is negligibly small, by a factor $\Delta T/(1 - \mathcal{T}_0)$. Thus within the softened layer, heat generation just balances diffusion. The advection becomes important on a much longer length scale, $O(h(1 - \mathcal{T}_0)/\Delta T)$, which in fact is large $O(\mathcal{K}^{1/(1-\alpha)})$. In this outer thermal boundary layer, there is no heat generation and the advection velocity is constant. The solution there, tending to the ambient \mathcal{T}_0 at great distances and tending to melting towards the softened layer $y \rightarrow 0+$, is therefore

$$T = \mathcal{T}_0 + (1 - \mathcal{T}_0)e^{-Pe|Vy|}.$$

This produces a temperature gradient just outside the softened layer

$$\left. \frac{\partial T}{\partial y} \right|_{0+} = -(1 - \mathcal{T}_0)PeV. \quad (18)$$

8 Solution for hard materials

We examine the details within the thin softened layer using the non-dimensionalisation of §5, and the rescalings just found (17) for the hard material limit. We introduce η as the rescaled distance across the layer in $y = \eta h$, and θ the rescaled temperature within the layer in $T = 1 - \theta\Delta T$. The modulus thus becomes $\kappa = \beta\mathcal{K}^{-(1+\alpha)/(1-\alpha)}\theta$. We now find the rescaled versions of the governing equations (11), (13) and (15).

The sliding velocity \tilde{u} is imposed and so is not rescaled. The shearing stress in the sliding motion is however rescaled to s in $S = s\beta\mathcal{K}^{-1/(1-\alpha)}$. (The shearing stress S is in fact small, because in the limit for hard materials the modulus is smaller than the shear rate is large.) The sliding equation (11) therefore becomes

$$\frac{\partial \tilde{u}}{\partial \eta} = s^{1/\alpha}\theta^{-1/\alpha}. \quad (19)$$

This is to be integrated from $\tilde{u}(0) = 0$ with unknown s selected so that $\tilde{u} \rightarrow 1$ as $\eta \rightarrow \infty$.

The squeezing velocity along the thins softened layer \bar{u} is rescaled to \bar{u}^* in $\bar{u} = \bar{u}^*(\alpha\beta)^{-1}$. The squeezing equation (13) becomes

$$\frac{\partial \bar{u}^*}{\partial \eta} = -3s^{(1-\alpha)/\alpha} \eta \theta^{-1/\alpha}. \quad (20)$$

The unknown $\bar{u}^*(0)$ is selected so that $\bar{u}^* \rightarrow 0$ as $\eta \rightarrow \infty$.

The squeezing velocity across the thins softened layer v is rescaled to v^* in $v = v^*(\alpha\beta)^{-1} \mathcal{K}^{-1/(1-\alpha)}$. The squeezing equation (14) becomes

$$\frac{\partial v^*}{\partial \eta} = -\bar{u}^*, \quad (21)$$

which is integrated from $v^*(0) = 0$ to find the rescaled velocity of approach of the work-pieces V^* in $v^* \rightarrow -V^*$ as $\eta \rightarrow \infty$.

As explained at the end of the previous section, the advection term in the heat equation (15) drops out in the thin softened layer. In terms of the rescaled variables, the equation becomes

$$\frac{\partial^2 \theta}{\partial \eta^2} = s^{(1+\alpha)/\alpha} \theta^{-1/\alpha}. \quad (22)$$

The matching condition (18) with the outer thermal boundary layer becomes, in these rescaled variables

$$\frac{\partial \theta}{\partial \eta} \rightarrow V^* \quad \text{as } \eta \rightarrow \infty. \quad (23)$$

The rescaled heat equation (22) is integrated from $\eta = 0$ where $\theta_\eta = 0$, and with the unknown $\theta(0)$ selected so as to satisfy the matching condition (23).

Numerical solution of the equations (19), (20), (21) and (22) for the case $\alpha = \frac{1}{4}$ are shown in figure 1. The values for the shooting parameters used are $s = 1.425$, $\bar{u}(0) = 1.046$, $\theta(0) = 1.245$. These give the scaled velocity of approach of the work-pieces $V^* = 1.425$.

Note that the integrals implied in the above differential equations converge if $\alpha < \frac{1}{3}$.

9 Conclusions for hard materials

Now assemble the above numerical results for $\alpha = \frac{1}{4}$, the rescalings of §7 and the original non-dimensionalisation of §5.

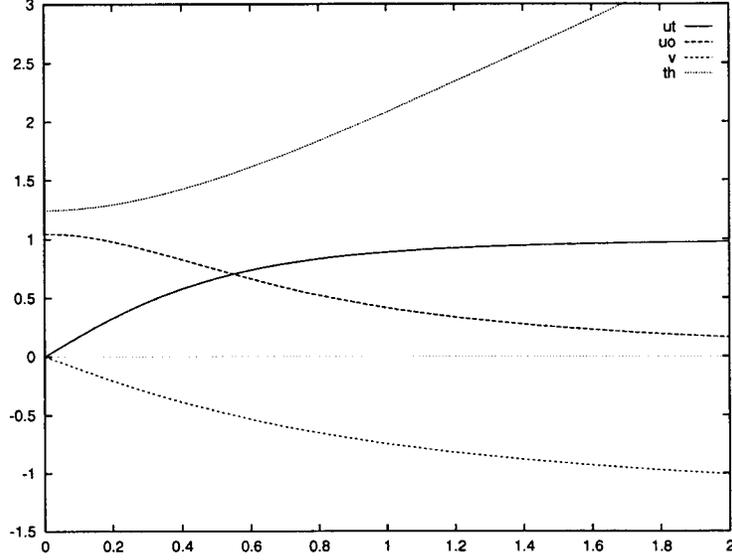


Figure 1: The profiles of the sliding velocity \tilde{u} (ut), squeezing velocity long the layer \bar{u} (uo), squeezing velocity across the layer v (v), and temperature θ (th), as functions of distance η across the thin layer.

The shear stress for the sliding motion was non-dimensionalised by PH/L . For hard materials it was rescaled by $\beta\mathcal{K}^{-4/3}$. The numerical value of the rescaled solution was $s = 1.425$. Thus

$$\sigma = 2.850(1 - T_0/T_m)^{1/2} \rho^{1/2} c_p^{1/2} T_m^{11/6} k^{4/3} \kappa^{-4/3} U_0^{-5/3} P^{1/2} L^{-1}.$$

For the example data, the value is 5.46×10^6 Pa.

The squeezing motion along the thin layer was non-dimensionalised by U_0 . For hard materials it was rescaled by $(\alpha\beta)^{-1}$. The numerical value of the rescaled solution was $\bar{u}(0) = 1.046$. Thus

$$\bar{u}(0) = 2.092(1 - T_0/T_m)^{-1/2} \rho^{-1/2} c_p^{-1/2} T_m^{-1/2} U_0 P^{1/2}.$$

For the example data, the value is $9.84 \times 10^{-2} \text{ m s}^{-1}$.

The squeezing motion across the thin layer was non-dimensionalised by $U_0 H/L$. For hard materials it was rescaled by $(\alpha\beta)^{-1} \mathcal{K}^{-4/3}$. The numerical value of the rescaled solution was $V^* = 1.425$. Thus

$$V = 2.850(1 - T_0/T_m)^{-1/2} \rho^{-1/2} c_p^{-1/2} T_m^{5/6} k^{4/3} \kappa^{-4/3} U_0^{-2/3} P^{1/2} L^{-1}.$$

For the example data, the value is $3.50 \times 10^{-4} \text{ m s}^{-1}$.

The thickness of the thin softened layer was non-dimensionalised by $H = \sqrt{\frac{kT_m L}{PU_0}}$. For hard materials it was rescaled by $\mathcal{K}^{-4/3}$. Thus

$$h = T_m^{4/3} k^{4/3} \kappa^{-4/3} U_0^{-5/3}.$$

For the example data, the value is 1.56×10^{-5} m.

The thermal boundary layer on the other hand was rescaled by $(\alpha\beta)\mathcal{K}^{4/3}Pe^{-1}$. The numerical value of the rescaled solution was $1/V^* = 0.702$. Thus its thickness is

$$0.351(1 - T_0/T_m)^{-1/2} \rho^{-1/2} c_p^{-1/2} T_m^{-5/6} k^{-1/3} \kappa^{4/3} U_0^{2/3} P^{-1/2} L.$$

For the example data, the value is 1.30×10^{-2} m.

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