Thermo-Electric Problem in the Søderberg Electrode

The Problem

The objective was to investigate the feasibility of using homogenization to calculate the temperature in a Søderberg electrode. In the region of interest (somewhat below the copper clamps used to feed current to the electrode) the general structure of the electrode is a cylindrical steel case of circular cross-section, with inward-pointing steel fins, and filled with carbon paste.

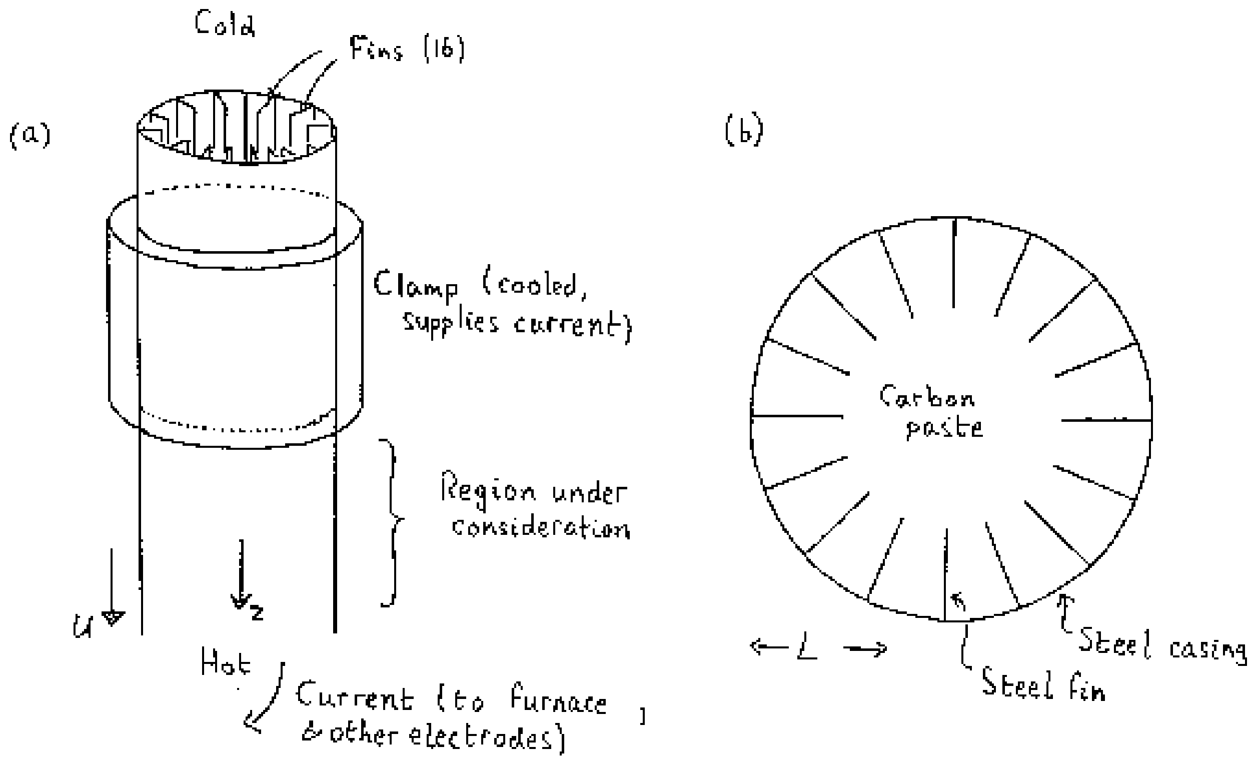

Figure 1: A Søderberg electrode. (a) Schematic diagram of the region under discussion. (b) Cross-section.

The electric current flows through the steel casing and fins, and through the paste, generating heat in each. There is heat transfer through advection, the electrode is moved downwards at 0.7 metres per day, as well as convection: heat is lost to the clamps which are water-cooled, radiative transfer takes place to and from the casing below the clamps; heat can also be supplied from the furnace below. To determine the temperature in the electrodes it is necessary to solve a coupled problem because the conductivity depends upon the temperature. Even in steady operation there is asymmetry in the electrode due to it being one of three in the furnace: higher temperatures are found in that part of the electrode closer to the furnace centre.

The questions to be answered were (i) did a simple homogenization method, averaging out the properties of the carbon and steel azimuthally (there are 16 fins inside the electrode), suffice to make good predictions? (ii) if not, how better might the calculations be carried out? For example, what would be a good way of positioning a mesh for a numerical solution?

In trying to answer (i) a nondimensional model of the coupled thermo-electrical problem was investigated taking direct-current operation. The true current is alternating and this was subsequently, but briefly, discussed.

The DC Model

For a slowly varying electric field the electric current $\mathbf{j}$ is given in terms of electric potential φ, $\mathbf{j} = -\sigma\nabla\varphi$ (σ = electrical conductivity), and, for charge neutrality, satisfies $\nabla\cdot\mathbf{j} = 0$. Heat is conducted, generated by Ohmic heating, and advected, so a steady temperature T satisfies $\nabla\cdot(\kappa\nabla T) + |\mathbf{j}\cdot\nabla\varphi| = U\rho C\frac{\partial T}{\partial z}$, taking z to be measured along the electrode in the direction of motion, *i.e.* downwards. (κ = thermal conductivity, U = electrode speed, ρ = density, C = specific heat.)

Hence φ and T satisfy

$$\nabla.(\sigma\nabla\varphi) = 0 \text{ and } \nabla.(\kappa\nabla T) + \sigma|\nabla\varphi|^2 = U\rho C\frac{\partial T}{\partial z}$$

in both the steel and the paste.

At steel/paste interfaces electric potential and temperature are continuous, while conservation of charge and heat hold:

$$[\varphi] = [T] = \left[\sigma\frac{\partial\varphi}{\partial n}\right] = \left[\kappa\frac{\partial T}{\partial n}\right] = 0,$$

for normal derivative $\frac{\partial}{\partial n}$.

Attention was focused on the electrode a little below the clamp. Here the outside of the electrode is electrically insulated but there is heat transfer due to thermal radiation:

$$\frac{\partial\varphi}{\partial n} = 0,\ \kappa\frac{\partial T}{\partial n} + \sigma_s T^4 = \sigma_s T_A^4.$$

(σ_s = Stefan's constant, T_A = external temperature.) (Further down, the steel disappears due to the extreme conditions in the furnace.)

In identifying large and small terms the following (typical) values were taken in the scaling. (Conductivities, for example, depended significantly upon temperature so here values around 800^{o}C - 1000^{o}C were taken; in the following normalized equations and conditions the temperature dependence, except that of σ in the paste, is suppressed; it can easily be reinstated.)

$$\Phi = \text{typical potential drop} = 1\text{V},\ U = 0.7\text{m day}^{-1} = 10^{-5}\text{m s}^{-1},$$

$$\rho_{steel} = 8\times 10^3 \text{kg m}^{-3}, \rho_{carbon} = 1.5\times 10^3\text{kg m}^{-3},$$

$$C_{steel} = 6\times 10^2 \text{J kg}^{-1}\text{K}^{-1}, C_{carbon} = 2\times 10^3\text{J kg}^{-1}\text{K}^{-1},$$

$$\kappa_{steel} = 30\text{W K}^{-1}\text{m}^{-1}, \kappa_{carbon} = 8\text{W K}^{-1}\text{ m}^{-1},$$

$$\sigma_{steel} = 6\times 10^5\text{m}^{-1}\Omega^{-1}, \sigma_{carbon} = 2.5\times 10^4\text{m}^{-1}\Omega^{-1},$$

$$L_s = \text{width of steel plate} = 3\times 10^{-3}\text{m},$$

L = macroscopic length of interest (both vertically and horizontally: the electrode radius and extent of the fin are comparable) = 0.5m,

$$T_A = 1150\text{K},\ \sigma_s = 6\times 10^{-8}\text{ W m}^{-2}\text{ K}^{-4}.$$

One easily noticeable small parameter is the ratio of lengths,

$$\epsilon \equiv L_s/L = 6\times 10^{-3}.$$

The equations in the steel are then

$$\nabla^2\varphi = 0,\ \nabla^2 T + \frac{\alpha}{\epsilon}|\nabla\varphi|^2 = Pe_s\frac{\partial T}{\partial z}$$

and those in the paste,

$$\nabla.(\sigma\nabla\varphi) = 0,\ \nabla^2 T + \beta\sigma|\nabla\varphi|^2 = Pe_c\frac{\partial T}{\partial z}, \tag{1}$$

with

$$\alpha \equiv \epsilon\sigma_{steel}\Phi^2/T_A\kappa_{steel} = 0.1, \quad Pe_s \equiv U\rho_{steel}C_{steel}/\kappa_{steel} = 1,$$

$$\beta \equiv \sigma_{carbon}\Phi^2/T_A\kappa_{carbon} = 3, \quad Pe_c \equiv U\rho_{carbon}C_{carbon}/\kappa_{carbon} = 2.$$

At their interfaces, $[\varphi] = [T] = 0$, $\frac{\partial\varphi}{\partial n}|_{steel} = \epsilon a\sigma\frac{\partial\varphi}{\partial n}|_{paste}$, $b\frac{\partial T}{\partial n}|_{steel} = \frac{\partial T}{\partial n}|_{paste}$, with $a \equiv \sigma_{carbon}/\epsilon\sigma_{steel} = 7$, $b \equiv \kappa_{steel}/\kappa_{paste} = 4$. At the outside boundary,

$$\frac{\partial\varphi}{\partial n} = 0,\ \frac{\partial T}{\partial n} + c(T^4 - 1) = 0$$

with $c \equiv L{T_A}^3\sigma_s/\kappa_{steel} = 1.5$.

In the steel sheet the normal co-ordinate can be rescaled (L_s is the appropriate distance across) so that with $X = 0$ on one side and 1 on the other,

$$\frac{\partial^2 \varphi}{\partial X^2} + \epsilon^2 \nabla_2^2 \varphi = 0, \qquad 0 < X < 1$$

$$\text{with } \frac{\partial \varphi}{\partial N} = \epsilon^2 a\sigma \left. \frac{\partial \varphi}{\partial n} \right|_{paste}$$

where $\frac{\partial}{\partial N}$ is the rescaled normal derivative in the steel and ∇_2 is the two-dimensional gradient using co-ordinates parallel to the plate. (If the plate is not flat, there is an extra term due to curvature but this has a relatively small effect.)

Then, to leading order, φ is independent of X and

$$a\sigma \frac{\partial \varphi}{\partial n} = \nabla_2^2 \varphi \text{ on the outside boundary of the paste,}$$

$$a\sigma \left[\frac{\partial \varphi}{\partial n} \right] + \nabla_2^2 \varphi = 0 \text{ across the fins.}$$

Likewise, for the temperature,

$$\frac{1}{\epsilon^2} \frac{\partial^2 T}{\partial X^2} + \nabla_2^2 T + \frac{\alpha}{\epsilon} (|\nabla_2 \varphi|^2 \ldots) = Pe_c \frac{\partial T}{\partial z}, \qquad 0 < X < 1,$$

$$\text{with } \frac{b}{\epsilon} \frac{\partial T}{\partial N} = \left. \frac{\partial T}{\partial n} \right|_{paste}, \quad \frac{\partial T}{\partial N} + \epsilon c (T^4 - 1) = 0 \text{ at the external surface.}$$

In this case,

$$\frac{\partial T}{\partial n} + bc\,(T^4 - 1) = \alpha b |\nabla_2 \varphi|^2 \text{for the outer boundary of the paste,}$$

$$\text{and } \left[\frac{\partial T}{\partial n} \right] + \alpha b |\nabla_2 \varphi|^2 = 0 \text{ across the fins.}$$

The problem has been reduced to a single pair of partial differential equations, (1), applying in the paste only, albeit one with somewhat unusual boundary (and internal jump) conditions.

Possible Homogenization

It was noted that although the number of fins is reasonably large, 16, the aspect ratio for each segment lying between neighbouring fins is close to unity, and the temperature can vary significantly between, as well as within, the segments (and there were quantities such as conductivities depending significantly upon temperature). There did not, therefore, seem a good prospect of achieving an accurate model based upon homogenization.

With a rather greater number of fins (say at least 50), occupying the same volume fraction (so the thickness of each is inversely proportional to the number) an averaged model could be possible.

The problem in the electrode, in dimensional variables, is

$$\nabla.(\sigma_c \nabla\varphi) = 0,\ \kappa_c \nabla^2 T + \sigma_c |\nabla\varphi|^2 = U\rho_c C_c \frac{\partial T}{\partial z} \tag{2}$$

away from the fins, with

$$\sigma_c \left[\frac{\partial\varphi}{\partial n}\right] + \sigma_s L_s \nabla_2^2 \varphi = 0,\ \kappa_c \left[\frac{\partial T}{\partial n}\right] + \sigma_s L_s |\nabla_2 \varphi|^2 = 0 \tag{3}$$

across them.

The following preliminary consideration of averaging supposes that an electrode has periodic structure with fins of width L_s, spaced at L_p in the y direction, so that $L_s \ll L_p \ll L$ (= length scale in the x (or r) and z directions as well as the macroscopic scale in the y direction). The y direction replaces the azimuthal (θ) direction in a real electrode (the fin spacing should depend upon the distance r from the axis of the electrode). A multiple-scales technique uses N = number of fins $\propto L/L_p \gg 1$, takes φ and T to depend upon the microscopic co-ordinate $Y = Ny$, as well as y and the other macroscopic space variables, and involves seeking T and φ as power series in N; for example $\varphi \sim \varphi_0 + \varphi_1/N + \varphi_2/N^2 \ldots$. Substituting these expressions into the modified equations (2) (where $\frac{\partial}{\partial y}$ is replaced by $\frac{\partial}{\partial y} + N\frac{\partial}{\partial Y}$) and jump conditions (3) (with $\left[\frac{\partial}{\partial n}\right]$ being replaced by $N\left[\frac{\partial}{\partial Y}\right]$) it is found, by comparing the largest terms, that to leading order φ and T are independent of Y (for example, $\varphi_0 = \varphi_0(x, y, z)$). The terms in 1/N then vanish while compatibility conditions from (2) and (3) with the requirement that φ_2 and T_2 be periodic as functions of Y (no secular terms) finally result in

$$\nabla.(\sigma_c \nabla\varphi_0) + \sigma_s (L_s/L_p) \nabla_2^2 \varphi_0 = 0,$$

$$\kappa_c \nabla^2 T_0 + \sigma_s (L_s/L_p) |\nabla_2^2 \varphi_0|^2 + \sigma_c |\nabla\varphi_0|^2 = U\rho C \frac{\partial T}{\partial z}.$$

The averaged equations for this idealised situation (laminar with parallel fins) are then, to leading order:

$$\nabla.(\sigma_c \nabla\varphi) + S\sigma_s \nabla_2^2 \varphi = 0, \kappa_c \nabla^2 T + \sigma_s S |\nabla_2^2 \varphi|^2 + \sigma_c |\nabla\varphi|^2 = U\rho C \frac{\partial T}{\partial z}$$

where $\nabla_2 = (\frac{\partial}{\partial x}, 0, \frac{\partial}{\partial z})$ is the two-dimensional gradient and $S = L_s/L_p$ is the volume fraction of steel. Note that these equations can be written as:

$$\nabla_2.(\sigma_p \nabla\varphi) + \frac{\partial}{\partial y}\left(\sigma_a \frac{\partial\varphi}{\partial y}\right) = 0, \kappa_c \nabla^2 T + \sigma_p |\nabla_2 \varphi|^2 + \sigma_a \left(\frac{\partial\varphi}{\partial y}\right)^2 = U\rho C \frac{\partial T}{\partial z}$$

where σ_p is the usual averaged conductivity, $= (1-S)\sigma_c + S\sigma_s \simeq \sigma_c + S\sigma_s$ since $S \ll 1$, and σ_a is given by averaging resistivities, $1/\sigma_a = (1-S)/\sigma_c + S/\sigma_s$ so $\sigma_a \simeq \sigma_c$ since $S \ll 1$

and $\sigma_c \ll \sigma_s$. These are the usual expressions for effective conductivities in the direction of lamenations and perpendicular to them. (Equivalent to finding resistances for parallel and series resistors respectively.) In the case of the thermal conductivities, those of the carbon and steel are comparable, so the effective values, both normal and parallel to the laminations, are approximately κ_c since the volume fraction of steel is small.

The AC Model

The truer problem of alternating current was looked at only very briefly. Two observations were made.

1. The conclusions about difficulties of homogenizing because of the order-one aspect ratio hold for alternating current as much as direct current.

2. Taking quantities such as current density to be written as $\mathrm{Re}\{\mathbf{j}e^{i\omega t}\}$, for angular frequency of the current ω, and neglecting electrical permittivity, $\mathbf{j}$ satisfies

$$\mu\omega\sigma\mathbf{j} = -i\nabla^2\mathbf{j}.$$

For $\omega = 300\mathrm{s}^{-1}$, permeabilities $\mu_{steel} = 9 \times 10^{-6}\mathrm{Hm}^{-1}, \mu_{paste} = 10^{-6}\mathrm{Hm}^{-1}$, the non-dimensional equation is:

$$\nabla^2\mathbf{j} = 400i\mathbf{j} \text{ in the steel, } \nabla^2\mathbf{j} = 2i\mathbf{j} \text{ in the carbon paste.}$$

Any substantial current penetration of the current into the electrode results from the significant paste skin depth.

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