MICROWAVING FOOD

1 Introduction

Microwaving is a common means of heating and cooking a foodstuff [2], [5] as well as being an important industrial process for heating materials such as ceramics. Whereas the physics of the latter process is well established (see [3]) and that of heating water easy to understand (see the winning entry of the Colemanballs competition), the processes of heating foodstuffs are as yet rather poorly understood. This is because the response of a foodstuff to the energy in a microwave field depends upon many factors (such as temperature and moisture content) which depend critically on the heating process [1]. As a result the literature on microwaving food is rather confused with poor agreement between theory and experiment. The objective of the Study Group was to obtain a model for heating a simple foodstuff such as a potato which allowed for the effects of a varying moisture content and which agreed more closely with experimental results than existing models.

2 Basic mechanisms

Consider a typical foodstuff indicated in Figure 1. In a microwave oven this is irradiated with microwaves at a frequency of 2.5 GHz (wavelength of 12cm). The electromagnetic waves penetrate the food, becoming attenuated as they do so. Heat is generated in the foodstuff by two mechanisms, ohmic heating and the excitation of dipoles. For the foods considered and the frequency of microwaves used, these two effects are roughly comparable. Heat is lost in the foodstuff through conduction, moisture transport and radiation at the surface.

3 Model for heating without moisture

Our initial model considered heating in the absence of moisture and largely follows work of Marchant and Smythe [?]. Consider a simple one dimensional material with an electromagnetic field

$$E(x,t) = E_0 e^{i\omega t} f(x) \tag{3.1}$$

incident from the left. Such a field satisfies

$$E_{tt} + (\sigma E)_t = c^2 E_{xx} \tag{3.2}$$

where c is the speed of light in the foodstuff and $\sigma(x,T)$ is the conductivity which depends upon postion x and temperature T. Suppose the foodstuff occupies the right hand plane $x \geqslant 0$ then

$$f_{xx} + \left(\frac{\omega^2 - i\omega\sigma}{c^2}\right) f = 0, \qquad |f| \to 0 \quad \text{as} \quad x \to \infty \tag{3.3}$$

so that

$$f_{xx} + \left(\frac{\omega}{c}\right)^2 \left(1 - \frac{i\sigma}{\omega}\right) f = 0 \tag{3.4}$$

If $\frac{\sigma}{\omega}$ is small and slowly varying then the WKB approximation gives

$$f \approx e^{\pm i\omega x/c} e^{\pm \int_0^x \sigma \, dx/2c} \tag{3.5}$$

so that the solution of interest is a rightward travelling wave of the form

$$E = E_0 e^{i\omega(t - \frac{x}{c})} e^{-\int_0^x \sigma \, dx/2c} \tag{3.6}$$

so that

$$|E| = E_0 e^{-\int_0^x \sigma \, dx/2c} \tag{3.7}$$

Observe that the wave is attenuated as it penetrates the foodstuff.

If σ is constant then

$$|E| = E_0 e^{-x\sigma/2c} = E_0 e^{-x/2\delta} \tag{3.8}$$

where $\delta \equiv c/\sigma$ is the *penetration depth* of the foodstuff. This parameter plays a vital role in the effectiveness of the cooking process. However its value is known very imprecisely and depends upon the foodstuff. For mashed potato, the value given by Dolande and Datta [2] is $\delta = 7$mm (implying surface heating only) whereas its value for water is $\delta = 3$cm ("so that if the potato were water, cooking it would be a piece of cake"). Thus δ depends significantly upon the moisture content of the food.

In the absence of moisture the microwaves heat the foodstuff with heat loss due to conduction so that

$$c_T \rho T_t = (kT_x)_x + Q(x,t) \tag{3.9}$$

where ρ is density, c_T specific heat, k the thermal conductivity and Q the heating effect of the microwaves. The form of Q is given by

$$Q = (\alpha\sigma + \beta\varepsilon'')|E|^2 \tag{3.10}$$

where ε'' is the dielectric loss factor and α, β are constants. (The two contributions to Q are due to ohmic heating and dipole polarisation respectively.) Simple estimates of c_T, ρ and k (see the other food sciences report showing how these can be determined) combined with some experiments on baking potatoes in an oven, indicate that a typical diffusion timescale for a potato is about one hour. Given that the time to microwave heat a potato is about 10 minutes, it is reasonable to ignore diffusion in our models. In the absence of a varying moisture content the values of α, β, σ and ε'' are roughly independent of temperature for a typical foodstuff. Thus we have simply that

$$\frac{dT}{dt} = (c_T\rho)^{-1}(\alpha\sigma + \beta\varepsilon'')E_0^2 \, e^{-\int_0^x \sigma \, dx/c} \tag{3.11}$$

Thus

$$T = T_0 + t(c_T\rho)^{-1}(\alpha\sigma + \beta\varepsilon'')E_0^2 \, e^{-\int_0^x \sigma \, dx/c} \tag{3.12}$$

where T_0 is the initial temperature.

For σ constant and a penetration depth of δ= 7mm this predicts that for a potato of radius 5cm, there is a factor of $e^{-50/7} = 7.9 \times 10^{-4}$ difference between the rate of heating at the surface and at the interior. In particular this predicts that the interior will not be heated in the time typically used to microwave the foodstuff. This is very different from what is observed in experiment indicating that additional processes are at work.

Studies of ceramics indicate that σ is strongly dependent upon T leading to thermal runaway and heat localisation reported in Kriegsmann [4]. However the literature on the response of foodstuffs [2] indicates that the electrical properties do not strongly depend upon temperature, ruling the above mechanism out as a means of enhancing the heating effect in the foodstuff. Of much greater significance is the variation of σ with moisture content.

4 Models which include moisture

4.1 Sophisticated models

We consider the following scenario. Microwaves striking the foodstuff heat up the surface first (due to the attenuation as they penetrate). This has the effect of driving off moisture with the temperature saturating at boiling point. This reduces the foodstuffs local conductivity and hence increases penetration depth. Thus more microwave energy reaches the centre, increasing heating there. Here we have used the fact that if $M(x,t)$ is the local moisture content, then σ is an increasing function of M.

A sophisticated model of this process would consider the production of steam in the foodstuff which then moves through the material under the action of a Darcy type law of transport. This ultimately would lead to a system requiring a solution of the porous medium equation with a nonlinear source term coupled to a Stefan problem and this was not attempted during the Study Group.

4.2 A simple free boundary model

For an intermediate model we consider a very simplified system where the steam produced on heating is allowed to escape and is not considered to affect the material either through a change in pressure or by heat transfer (see later for a discussion of this approximation). The latter assumption comes from the observation that the steam will mainly move outward from the heated areas and will thus not affect the model for heating the centre. The pressure change is a more significant change from observation. Witness the fact that potatoes baked in a microwave oven can explode! We return to this presently. For our model we take T_b to be the boiling temperature and make the following assumptions:

1. There is a free boundary at $x = s > 0$ such that if the surface of the foodstuff is at $x = 0$ then

$$T = T_b \quad \text{if} \quad x \leqslant s \quad \text{and} \quad T < T_b \quad \text{if} \quad x > s. \tag{4.1}$$

2. In the centre of the foodstuff $T < T_b$. The moisture content of the food does not change and all of the microwave energy is used to heat the food.

3. If $x < s$ then all of the microwave energy is used to reduce the moisture content.

4. Initially the foodstuff has a uniform temperature $T < T_b$ and a uniform moisture content M.

The heating effect of the microwaves is then given by a term proportional to $\sigma(M)|E|^2$ which in general is given by

$$Q = \sigma(M) \exp\left(-\int_0^x \sigma(M) dx/c\right)$$

So, under these assumptions we have the following model

• If $x < s$ then

$$T = T_b, \quad T_t = 0, \quad M_t = -\beta\sigma(M)\exp\left(-\int_0^x \sigma(M) dx/c\right). \tag{4.2}$$

• If $x > s$ then

$$T < T_b, \quad T_t = \alpha\sigma(M)\exp\left(-\int_0^x \sigma(M) dx/c\right), \quad M_t = 0. \tag{4.3}$$

Here α, β are appropriate constants estimated from the thermal properties of the foodstuff.

The position s of the free boundary at which the moisture boils is given by the identity

$$T(s,t) = T_b, \tag{4.4}$$

and for initial data we take

$$T(x,0) = T_0 < T_b \quad M(x,0) = M_0. \tag{4.5}$$

It is unclear what the precise form taken by $\sigma(M)$ is, so for (drastic) simplicity we hypothesise that

$$\sigma(M) = \sigma M \tag{4.6}$$

Now make the following non-dimensionalisations

$$\widehat{T} = \frac{T}{T_b}, \quad \widehat{M} = \frac{M}{M_0}, \quad \widehat{x} = \frac{\sigma M_0 x}{c}, \quad \widehat{t} = \frac{(\alpha\sigma M_0)}{T_b}t$$

then rescaling and dropping the 'hats' for clarity we get

• If $x < s$ then

$$T = 1, \quad M_t = -\theta M \exp(-\int_0^x M dx), \quad M < 1. \tag{4.7}$$

• If $x > s$ then

$$T < 1, \quad T_t = M \exp(-\int_0^x M dx), \quad M_t = 0. \tag{4.8}$$

Here

$$\theta = \beta T_b/(\alpha\sigma M_0) \tag{4.9}$$

with

$$T = T_0/T_b \equiv \gamma < 1 \quad \text{and} \quad M = 1 \quad \text{initially} \tag{4.10}$$

This intermediate model has an analytic solution and a typical distribution of $T(x,t), M(x,t)$ is indicated below, with the free boundary at s.

We now show that asymptotically the solution comprises a *travelling wave* which moves into the foodstuff at a non-dimensional speed K given by

$$K = \theta/(1+(1-\gamma)\theta) \tag{4.11}$$

Observe that this implies a much more rapid increase in the central temperature than the model without moisture.

Initially $T < 1$ and $M = 1$ everywhere. There is then an initial phase where the moisture content does not change and $s \equiv 0$. This continues until $t = 1-\gamma$ when $T(0, 1-\gamma) = 1$. During this phase

$$T(x,t) = \gamma + te^{-x} \tag{4.12}$$

At $t = 1-\gamma$ the free boundary starts to move into the foodstuff. Differentiating the identity $T(s,t) = 1$ we have

$$T_x \frac{ds}{dt} = -T_t \qquad \text{at} \qquad x = s^+. \tag{4.13}$$

When $t = (1-\gamma)$ we have from (4.12) that $T_t = 1$, $T_x = -(1-\gamma)$ so that initially

$$\frac{ds}{dt} = \frac{1}{1-\gamma}. \tag{4.14}$$

Let $I(x,t) = \int_0^x M\,dx$, then

$$M_t = -\theta M e^{-I} \quad \text{so that} \quad I_{xt} = -\theta e^{-I} I_x. \tag{4.15}$$

Integrating and setting $I(0,t) = 0$ gives

$$I_t = \theta(e^{-I} - 1). \tag{4.16}$$

We now consider how the front at $x = s$ propogates into the material by reducing the description of the front motion to an ordinary differential equation problem. Let

$$J(t) = \int_0^s M\,dx \equiv I(s,t). \tag{4.17}$$

Then

$$\dot{J} = I_t(s,t) + M(s,t)\dot{s} = \theta(e^{-J} - 1) + \dot{s}. \tag{4.18}$$

Now, if $x > s$, $M = 1$ so that

$$T_t = Me^{-\int_0^x M\,dx} = e^{-\int_0^x M\,dx} = e^{-J} e^{s-x}. \tag{4.19}$$

Thus

$$T = \gamma + e^{-x} \int_0^t e^{-J} e^s dt, \quad T_x = -e^{-x} \int_0^t e^{-J} e^s dt, \tag{4.20}$$

so that

$$\dot{s} = -T_t/T_x = e^s e^{-J} / \int_0^t e^{-J} e^s dt.$$

Putting $u(t) = s(t) - J(t)$ this simplifies to

$$\dot{s} = e^u / \int_0^t e^u dt, \qquad \dot{u} = \theta(1 - e^{u-s}). \tag{4.21}$$

It is convenient to introduce a new variable

$$\tau = t - (1 - \gamma) \tag{4.22}$$

so that $\tau = 0, u = 0, s = 0$ when the front starts to move into the foodstuff. We note that

$$\int_0^{1-\gamma} e^u dt = 1 - \gamma. \tag{4.23}$$

Then expressing derivatives with respect to τ

$$\dot{s} = e^u / (1 - \gamma + \int_0^\tau e^u d\tau) \tag{4.24}$$

$$\dot{u} = \theta(1 - e^{u-s}) \tag{4.25}$$

This system has an exact solution and in the limit of $t \to \infty$ we have

$$s/t \to K \quad \text{and} \quad J \to J_\infty \tag{4.26}$$

where

$$K = \theta/(1 + (1 - \gamma)\theta) \tag{4.27}$$

$$J_\infty = \log((1 + (1 - \gamma)\theta)/(1 - \gamma)\theta). \tag{4.28}$$

To establish this result we differentiate (4.24) with respect to τ to give

$$\ddot{s} = \dot{s}\dot{u} - \dot{s}^2. \tag{4.29}$$

Thus

$$\frac{\ddot{s}}{\dot{s}} = \dot{u} - \dot{s} \tag{4.30}$$

so that on integrating and using (4.14) we have

$$\dot{s} = Ae^{u-s} = \frac{e^{u-s}}{1 - \gamma}. \tag{4.31}$$

Rearranging and substituting the formula for $\dot{u}$ gives

$$\dot{u} = \theta(1 - (1 - \gamma)\dot{s}) \tag{4.32}$$

so that as $u = s = 0$ at $\tau = 0$

$$u = \theta\tau - (1 - \gamma)\theta s. \tag{4.33}$$

Substituting we have

$$\dot{s} = \frac{1}{1-\gamma} e^{\theta\tau} e^{-(1+(1-\gamma)\theta)s} \tag{4.34}$$

so that integrating and applying the initial conditions

$$\frac{e^{(1+(1-\gamma)\theta)s}}{1+(1-\gamma)\theta} = \frac{e^{\theta\tau}}{\theta(1-\gamma)} + \frac{1}{1+(1-\gamma)\theta} - \frac{1}{\theta(1-\gamma)} \tag{4.35}$$

and hence

$$e^{(1+(1-\gamma)\theta)s} = \frac{(1+(1-\gamma)\theta)e^{\theta\tau}}{\theta(1-\gamma)} - \frac{1}{\theta(1-\gamma)}. \tag{4.36}$$

Thus, the general expressions for s and J are

$$s = \frac{1}{1+(1-\gamma)\theta} \log\left(\frac{1}{\theta(1-\gamma)}(e^{\theta\tau}(1+(1-\gamma)\theta) - 1)\right), \tag{4.37}$$

$$J = \log\left(\frac{1}{\theta(1-\gamma)}((1+(1-\gamma)\theta) - e^{-\theta\tau})\right). \tag{4.38}$$

For large τ we get

$$s \approx \frac{\theta\tau}{1+(1-\gamma)\theta} + \frac{1}{1+(1-\gamma)\theta} \log\left(\frac{1+(1-\gamma)\theta}{\theta(1-\gamma)}\right) \tag{4.39}$$

and

$$J \approx \log\left(\frac{1+(1-\gamma)\theta}{\theta(1-\gamma)}\right) \tag{4.40}$$

as required.

This result implies that the free boundary moves into the foodstuff at an asymptotically constant speed K.

We see this from the following plot of s when $\theta = 5, \gamma = 1/2$.

We now calculate T and M.

From (4.19), (4.31) we have that if $x > s$ then

$$T_t = e^{-x}e^{u} = (1-\gamma)e^{-x}e^{s}\dot{s}$$

Integrating and setting $T(s,t) = 1$ we have simply that if $x > s$

$$T(x,t) = \gamma + (1-\gamma)e^{s-x}. \tag{4.41}$$

Observe this predicts a simple thermal wave moving into the food stuff at an asymptotically constant speed.

Integrating (4.16), there is a function $g(x)$ such that

$$I(x,t) = \log(1 + e^{-\theta t} g(x))$$

and by definition

$$I(s,t) = J(t).$$

Given a fixed x there is a time $t(x)$ at which $s(t(x)) = x$. Then

$$I(x, t(x)) = \log(1 + e^{-\theta t(x)} g(x)) = J(t(x))$$

so that

$$g(x) = e^{\theta t(x)} (e^{J(t(x))} - 1).$$

Hence, if $t > t(x)$ then

$$I(x, t) = \log(1 + e^{\theta(t(x) - t)} (e^{J(t(x))} - 1)).$$

As s and J are both given explicitly, this gives $I(x, t)$ and hence $M(x, t)$. Asymptotically, if x and t are large

$$t(x) \approx x/K \quad \text{and} \quad e^J - 1 \approx \frac{1}{\theta(1-\gamma)}$$

so that

$$I(x, t) \approx \log(1 + \frac{e^{\theta(x/K - t)}}{\theta(1-\gamma)})$$

and hence

$$M(x, t) \approx \frac{e^{\theta(x/K - t)}}{K(1-\gamma)(1 + \frac{e^{\theta(x/K-t)}}{\theta(1-\gamma)})}$$

observing that $M(s, t) = 1$ as required.

5 Some comments on the model

The above model predicts a thermal wave moving into the foodstuff and heating it more quickly than a model without moisture. However it has made the assumption that the foodstuff is a very open porous material in which moisture can escape easily. For a potato (with a skin) it is certainly not clear that the moisture will escape. A perhaps more appropriate model would be one in which the moisture redistributes through the whole material and condenses. This gives an additional heating mechanism for the interior of the food. There is a lot of room here for additional modelling and some experiments (perhaps as a prelude for supper).

References

[1] Dealler, S., Rotawa, N. and Lacey, R., "Ionized molecules reduce penetration of microwaves into food", Int. J. of Food Science and Technology **27**, (1992), 153-157.

[2] Dolande, J. and Datta, A., "Temperature profiles in microwave heating of solids: a systematic study", J. of Microwave Power and Electromagnetic Energy, **28**, (1993), 58-67.

[3] Kriegsmann G., Brodwin, M. and Watters, D. "Microwave heating of a ceramic halfspace", SIAM J. Appl. Math., **50**, (1990), 1088-1098.

[4] Kriegsmann, G. "Thermal runaway in microwave heated ceramics: a 1-D model", J. Appl. Phys. **71**, (1992), 1960-1966.

[5] Marchant, T. and Smythe, N. " Microwave heating of materials with nonohmic conductance", SIAM J. Appl. Math. **53**, (1993), 1591-1612.

[6] R. Mudgett, "Microwave properties and heating characteristics of foods", Food Technology, June (1986), 84-93.

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