# BUCKLING OF ROLLED STRIP

## 1. Introduction

When a strip of metal formed by cold rolling is examined, it is frequently found to contain buckles. If centre buckles occur, the strip is said to have a 'long centre' while edge buckles are usually attributed to 'long edges'. If the wavelength and amplitude of the buckles are large, the shape of the buckles will obviously depend on the configuration in which the strip is held. For example, we expect the pattern of buckles of a strip laid on a flat table to be different those of a strip suspended between two rollers and subjected to tension. It is observed in practice, however, that buckling occurs in some cases with a characteristic wavelength that is relatively robust to strip configuration.

Clearly an understanding of buckling patterns is of considerable interest to manufacturers of strip, particularly as current trends are to produce thinner sheet which is consequently more prone to buckling. The specific question that BHP Coated Products Division asked the Study Group to consider was whether long centres and edges could give rise to buckling patterns with a robust characteristic frequency.

The question of buckling in strip has been addressed by a number of authors (*e.g.* Ishikawa *et al.* (1987), Jiachuang *et al.* (1987), Kuhne & Magdeburg (1975), Nomato *et al.* (1987)). In most cases these papers deal with the onset of instability in the strip and only Jiachuang *et al.* examine postbuckling behaviour (using a finite element calculation). Because of the constraints of the Study Group, we have only attempted to examine the onset of instability. Even for the simplest case (an eigenvalue problem for a fourth order differential equation with constant coefficients), the algebra becomes unmanageable and we have therefore resorted to some rather crude and limited numerical calculations. Furthermore, we have speculated about the lowest energy states of some buckling modes on the basis of intuition alone and clearly further work is required to verify our tentative conclusions.

## 2. Preliminaries

We consider an infinite strip as in Figure 1. Let us assume that when the strip is flat it is in equilibrium (albeit possibly unstable) and that the residual stresses $\sigma_x^{(0)}$, $\sigma_y^{(0)}$, $\tau^{(0)}$, say, in this configuration are independent of $x$. From equilibrium,

$$\frac{\partial \sigma_x^{(0)}}{\partial x} + \frac{\partial \tau^{(0)}}{\partial y} = \frac{\partial \tau^{(0)}}{\partial y} = 0$$

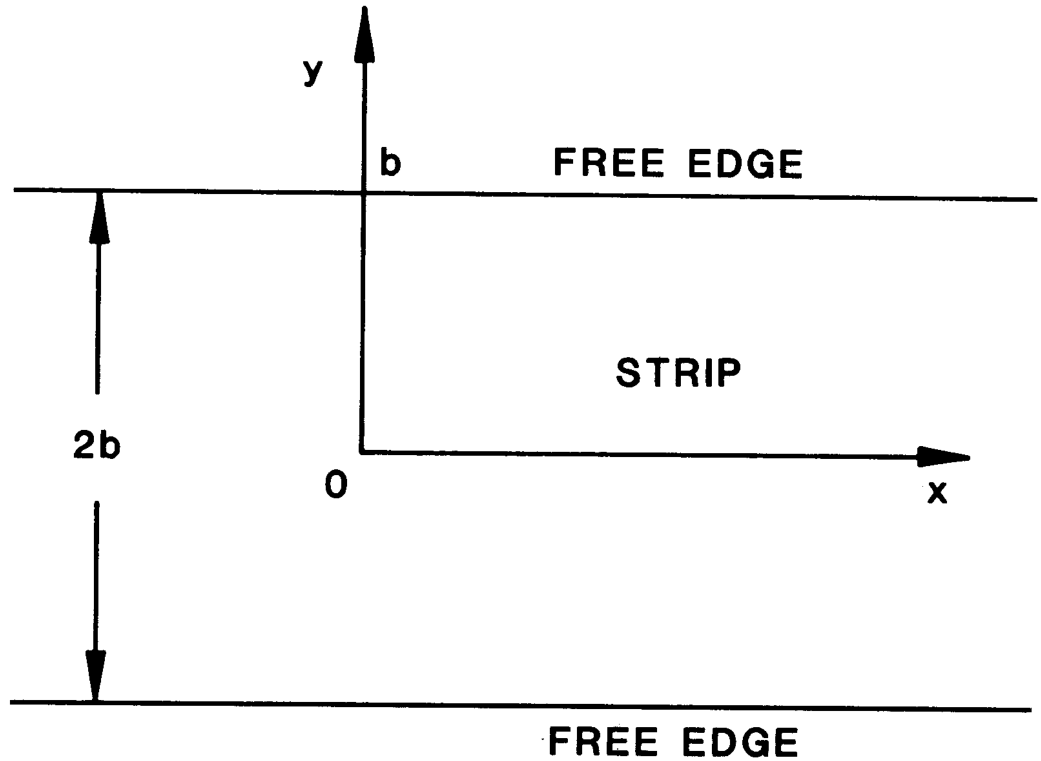


Figure 1: Geometry of strip.

and

$$\frac{\partial \sigma_y^{(0)}}{\partial y} + \frac{\partial \tau^{(0)}}{\partial x} = \frac{\partial \sigma_y^{(0)}}{\partial y} = 0$$

together with the boundary conditions

$$\tau^{(0)} = \sigma_y^{(0)} = 0, \quad y = \mp b,$$

we obtain

$$\tau^{(0)} = \sigma_y^{(0)} = 0, \quad -b \le y \le b,$$
$$\sigma_x^{(0)} = \sigma_x^{(0)}(y), \quad -b \le y \le b.$$

The situation when the above equilibrium state becomes unstable is more complicated. From Hooke's law, we have

$$\bar{\epsilon}_x = \frac{1}{E}[\bar{\sigma}_x - \sigma_x^{(0)} - \nu\bar{\sigma}_y], \tag{1}$$

$$\bar{\epsilon}_y = \frac{1}{E}[\bar{\sigma}_y - \nu(\bar{\sigma}_x - \sigma_x^{(0)})], \tag{2}$$

$$\bar{\gamma} = \frac{2(1+\nu)}{E}\bar{\tau} \tag{3}$$

where $\bar{\sigma}_x$, $\bar{\sigma}_y$, $\bar{\tau}$ and $\bar{\epsilon}_x$, $\bar{\epsilon}_y$, $\bar{\gamma}$ are the averaged stress and strains respectively. (Note that the average refers to averaging through the thickness of the strip.) The parameters $E$ and $\nu$ are the Young's modulus and Poisson ratio respectively. The strains are given by

$$\bar{\epsilon}_x = \frac{\partial u}{\partial x} + \frac{1}{2}(\frac{\partial w}{\partial x})^2, \tag{4}$$

$$\bar{\epsilon}_y = \frac{\partial v}{\partial y} + \frac{1}{2}(\frac{\partial w}{\partial y})^2, \tag{5}$$

$$\bar{\gamma} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{6}$$

where $u$, $v$ and $w$ are the longitudinal, lateral and vertical displacements of the neutral plane of the strip. Equilibrium of the strip requires

$$\frac{\partial \bar{\sigma}_x}{\partial x} + \frac{\partial \bar{\tau}}{\partial y} = 0, \tag{7}$$

$$\frac{\partial \bar{\sigma}_y}{\partial y} + \frac{\partial \bar{\tau}}{\partial x} = 0, \tag{8}$$

$$D\Delta^2 w = h\bar{\sigma}_x\frac{\partial^2 w}{\partial x^2} + 2h\bar{\tau}\frac{\partial^2 w}{\partial x \partial y} + h\bar{\sigma}_y\frac{\partial^2 w}{\partial y^2} \tag{9}$$

where $D = h^3/12(1-\nu^2)$, $h$ is the sheet thickness and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Equations (1-9) need to be augmented by some boundary conditions. We shall be looking for a periodic solution in $x$ with wavelength $2\pi/f$ ($f$ is unknown). Furthermore, as the edges $y = \mp b$ are not constrained, we require

$$\bar{\sigma}_y = \bar{\tau}, \quad y = \mp b, \tag{10}$$

$$\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2} = 0, \quad y = \mp b, \tag{11}$$

$$\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y} = 0, \quad y = \mp b. \tag{12}$$

## 3. Scaling and linearization

In order to reduce the number of parameters, it is worthwhile scaling the above equations. An obvious scaling for the length is $b$, the halfwidth of the strip, while a natural scale for the stress is $E$, the Young's modulus. However, in order to avoid introducing additional notation for the scaled variables, we shall assume that we are working in units for which $b = 1$, $E = 1$.

It is easy to verify that a solution of (1-12) is $u, v, w = 0$, $\bar{\sigma}_y = \bar{\tau} = 0$, $\bar{\sigma}_x = \sigma_x^{(0)}$. Furthermore, it will be an isolated solution unless the linearized problem has a nontrivial solution. That is, unless

$$\Delta^2 w = \frac{h}{D}\sigma_x^{(0)}\frac{\partial^2 w}{\partial x^2}$$

subject to the boundary conditions.

$$\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2} = \frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y} = 0, \quad y = \mp 1,$$

has a nontrivial solution.

If we look for a solution of the form

$$w = a \cos(fx)\, W(y),$$

we obtain

$$[\frac{d^2}{dy^2} - f^2]^2 W = -\frac{hf^2}{D}\sigma_x^{(0)} W, \tag{13}$$

$$W'' - \nu f^2 W = W''' - (2-\nu) f^2 W' = 0, \quad y = \mp 1, \tag{14}$$

$$\int_{-1}^{1} W^2(y)dy = 1. \tag{15}$$

Depending on the form of $\sigma_x^{(0)} = \sigma_x^{(0)}(y)$, there may exist one or more values of $f$ for which (13,14) have a nontrivial solution. Nevertheless, such value(s) do not necessarily correspond to a characteristic frequency for buckles in the strip.

To progress further, let us write

$$\frac{h\sigma_x^{(0)}}{D} = T - Pg(y)$$

where

$$\int_{-1}^{1} g(y)dy = 0, \quad T \geq 0.$$

Clearly, $T$ is a parameter associated with tension in the sheet while $P$ is a parameter associated with distortion due to the rolling process. For our purposes, it is convenient to take a simple form of $g$ such as

$$g(y) = 1 - (\alpha + 1) \mid y \mid^{\alpha}$$

or

$$g(y) = \begin{cases} 1 & |y| < \beta \\ -\beta/(1-\beta) & |y| > \beta \end{cases} \tag{16}$$

A representation such as (16) is particularly attractive from an analytical point of view as the fundamental solutions of (13) are easily calculable in the intervals $|y| < \beta$, $|y| > \beta$. However the algebra required to match $W$, $W'$, $W''$, $W'''$ at $|y| = \beta$ is daunting even if symmetry is assumed.

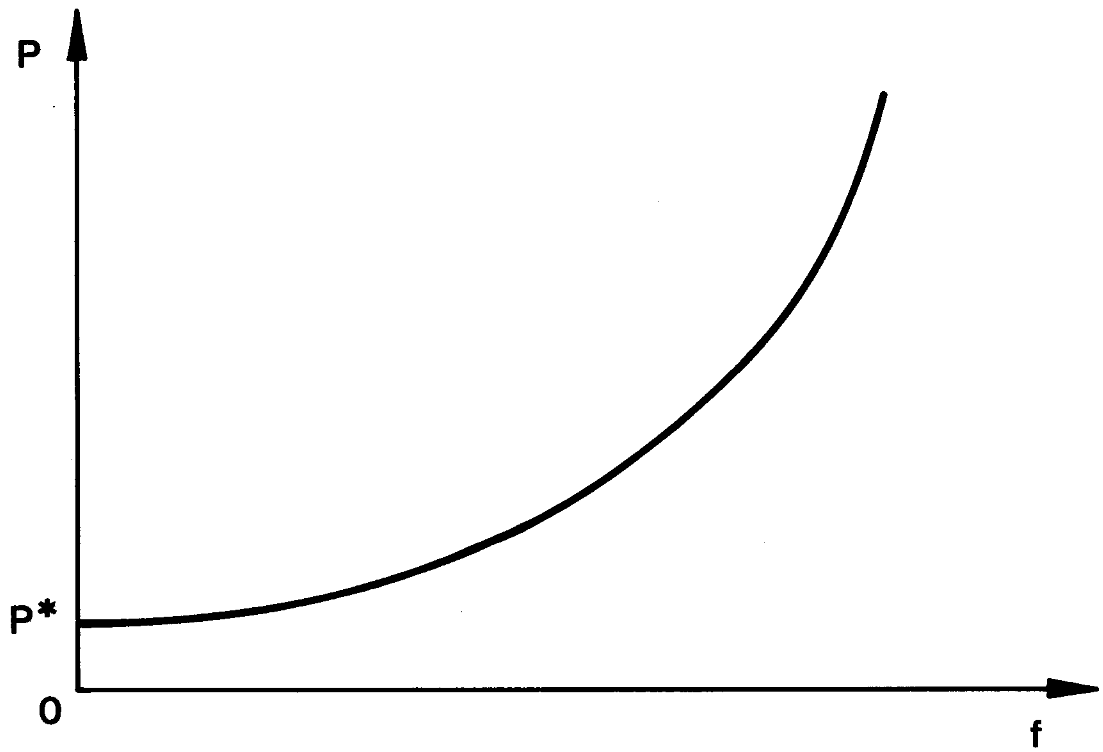


Figure 2: Centre buckling when $T = 0$.

Let us first consider the case when $T = 0$. We have at our disposal two parameters $P$ and $f$ that can be selected so that equations (13-15) have a nontrivial solution. In general we obtain one parameter families of curves such as the one shown in Figure 2 which corresponds to centre buckling.

Figure 2 suggests that there is a value $P^*$ such that (13-15) (with $T = 0$) have only the trivial solution when $P < P^*$. Because this value corresponds to $f = 0$, it is possible to calculate it (in principle) by approximating the fundamental solutions of (13-15) when $f$ is small by a regular perturbation expansion. The expansion is calculable using quadrature. For example, if

$$g(y) = 1 - 3x^2,$$

a relatively straightforward but tedious calculation yields

$$P^* = \{\sqrt{315/2 - 189\nu^2/4} + 21\nu/2\}/2.$$

Hence, for $\nu = 0.3$, we obtain

$$P^* \doteq 7.765$$

which is consistent with results of Magdeburg & Kuhne (1975) where the value was calculated by solving (13-15) numerically. In practice however, $P > P^*$, in which case buckling does occur although the relationship between $P$ and the frequency of the buckle is unlikely to correspond to Figure 2.

Let us now consider the case where $T \neq 0$. As previously, for a fixed value of $T > 0$, we obtain one parameter families of curves, an example of which is shown in Figure 3.

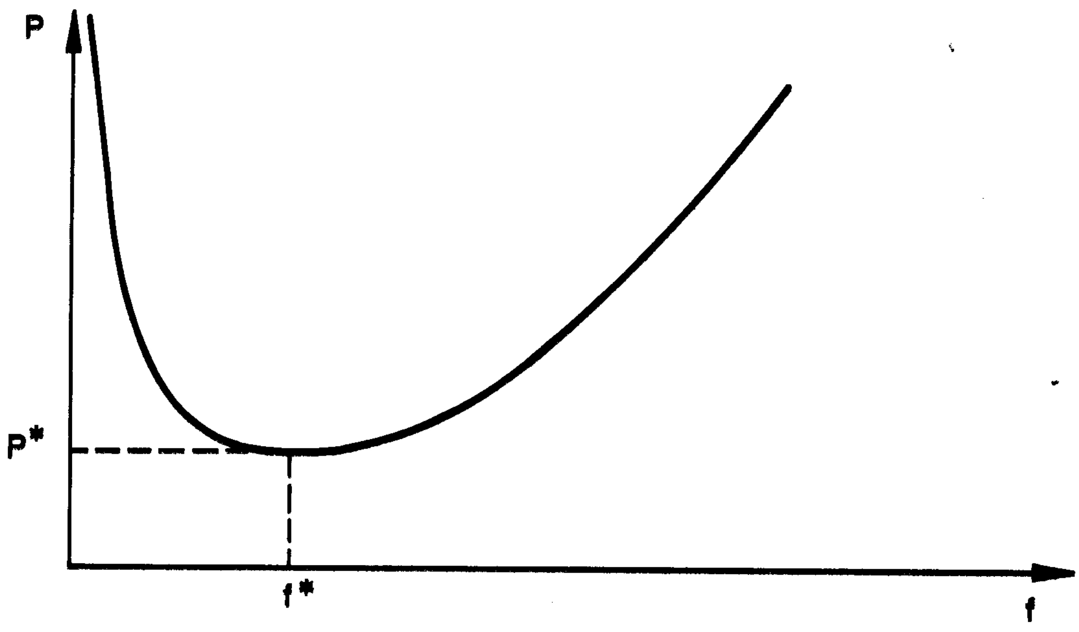


Figure 3: Centre buckling when $T > 0$

A feature of Figure 3 is a unique minimum at the point $(f^*, P^*)$. This point obviously depends on $T$. That is,

$$f^* = f(T), \quad P^* = P(T)$$

and we find from our previous consideration of $T = 0$ that $f(0) = 0$.

A typical profile of $P(T)$ is shown in Figure 4. Note that $P(T)$ is almost linear in $T$ for large $T$. The curve $P(T)$ may be thought of as the interface between buckled and unbuckled strip. That is, a strip with parameters $P$, $T$ will be buckled if $P > P(T)$. An additional property we shall find useful is

$$P'(T) = 1/\int_{-1}^{1} W^2(y)g(y)dy.$$

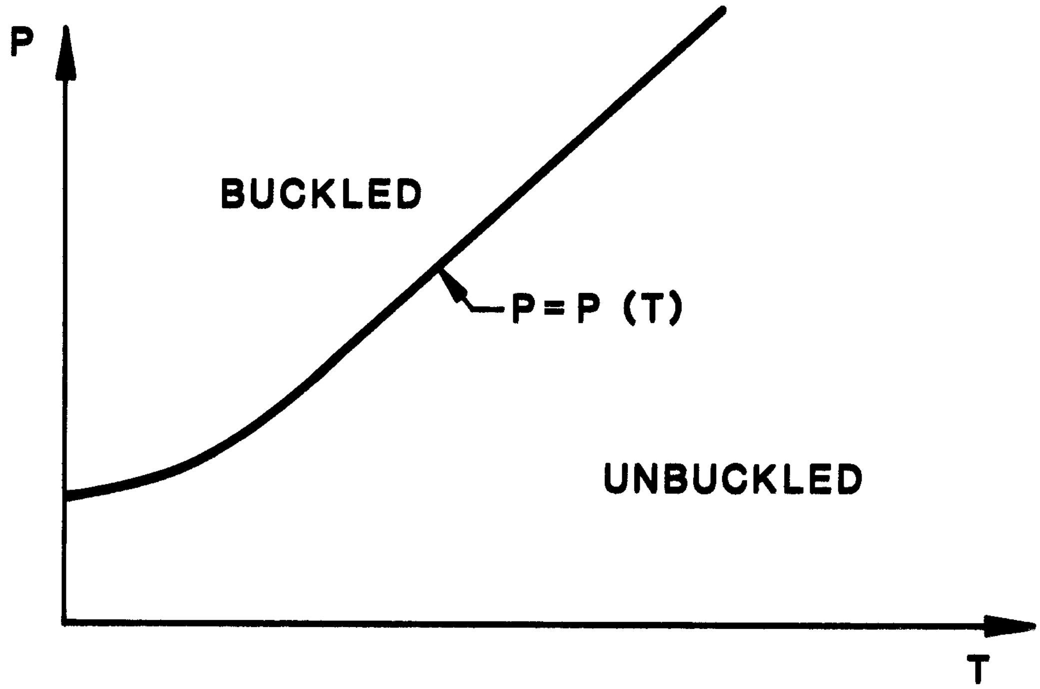


Figure 4: $P(T)$ *versus* $T$ for centre buckling.

## 4. Perturbation solutions

When the amplitude of the vertical displacement is small, it is possible to develop a perturbation solution to the problem outlined in the previous section. We have developed the first term in the expansion but this is insufficient to examine the question of characteristic frequencies. However, the approach taken does, in principle, generalise so that this question could be examined.

Let us suppose that we have a strip with parameters $P^*$, $T^*$ near the curve $P = P(T)$. Then we may assume

$$\begin{aligned} w &= a \cos(fy)\, W(y) + O(a^2), \\ \bar{\sigma}_x &= \sigma_x^{(0)} + a^2\{A(y) + B(y)\cos(2fx)\} + O(a^4), \\ \bar{\sigma}_y &= a^2 C(y)\cos(2fx) + O(a^4), \\ \bar{\tau} &= a^2 F(y)\sin(2fx) + O(a^4) \end{aligned}$$

where $f = f(T)$ and $T \sim T^*$. If we could calculate these terms and the next order terms, it would be possible to minimize the strain energy with respect to $T$ and thus obtain a frequency $f(T)$. Unfortunately the expression for strain energy obtained in this way is rather complicated and obviously requires numerical work to resolve the minimum. We have therefore confined attention to the leading order term.

From the equilibrium equations (7,8)

$$F = -\frac{1}{2f}C',$$

$$B = -\frac{1}{4f^2}C''.$$

On substituting for $F$ and $B$ in the compatibility equation

$$\frac{\partial^2}{\partial y^2}(\bar{\sigma}_x - \sigma_x^{(0)}) - 2\frac{\partial^2 \tau}{\partial x \partial y} + \frac{\partial^2 \bar{\sigma}_y}{\partial x^2} = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2},$$

we obtain

$$\left(\frac{d^2}{dy^2} - 4f^2\right)^2 C = \frac{2w^4 h}{D}(W'^2 - WW''),$$

$$C = C' = 0, \quad y = \mp 1$$

and

$$A = \frac{hf^2}{4D}\left(W^2 - \frac{1}{2}\right).$$

If we now substitute for $w$, $\bar{\sigma}_x$, $\bar{\sigma}_y$, $\bar{\tau}$ into equation (9) we find, after some algebra,

$$-f^2[\Delta T - \Delta P/P'] = \frac{A^2 D}{h}\left[\int_{-1}^{1}\{4A^2 + \frac{1}{4f^4}\left(\frac{d^2 C}{dy^2} - 4f^2 C\right)^2\}dy\right] \qquad (17)$$

where

$$\Delta T = T - T^*, \quad \Delta P = P(T) - P^*.$$

Since we are working to first order,

$$\Delta T \;-\; \Delta P/P' \;\sim\; -(T^* - T)$$

where $P(T^*) = P^*$. Hence $a^2$ is proportional to $T^* - T$.

The strain energy per unit length is

$$\mathcal{E} \;=\; \frac{f}{2\pi}\int_{-\pi/f}^{\pi/f}\int_{-1}^{1}[\frac{h}{2}\{\bar{\sigma}_x^2 + \bar{\sigma}_y^2 - 2\nu\bar{\sigma}_x\bar{\sigma}_y + 2(1+\nu)\bar{\tau}^2\}$$

$$+\;\frac{D}{2}\{(\frac{\partial^2 w}{\partial x^2})^2 + (\frac{\partial^2 w}{\partial y^2})^2 + 2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)(\frac{\partial^2 w}{\partial x\partial y})^2\}]dxdy.$$

To first order, the expansion reduces to

$$\mathcal{E} \;\simeq\; \mathcal{E}^{(0)} \;-\; \frac{a^2 D}{4} f^2 T$$

which shows that the strain energy of the buckled state is lower than the unbuckled state. However, higher order terms are necessary to obtain the value of $T$ for which the strain energy is minimized.

## 5. Some speculations

It is clear that there are a wide variety of buckled states and this is due to the fact that we have modelled the strip as an infinite one and consequently the spectrum of the linearized equation is continuous rather than discrete. Presumably the buckled state is that of lowest strain energy but an investigation of that requires further analytical work.

During the Study Group some analogous problems were identified that bear some similarities to the problem in hand, but these analogues yield conflicting results. On physical grounds, it was felt that if a strip was held at a tension $T$ with $P(T) \leq P$ and the tension was slowly decreased, then the frequency at which the buckle first appears is the characteristic buckle. That is, if $P(T^*) = P$, then the characteristic buckle has a frequency $f(T^*)$. There is however, no mathematical basis for this conjecture and obviously further work is required.

In conclusion, the moderator for this problem (Frank de Hoog) would like to acknowledge the contributions to the discussions by Archie Brown, Barrie Fraser, Noel Thompson, Peter Trudinger, John van der Hoek, Hugh Williamson and Daniel Yuen.

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