OPTIMISING THE RELATIONSHIP OF ELECTRICITY SPOT PRICES TO INPUT DATA

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Abstract

Electrical power generation is paid for on the basis of the marginal costs of generation, and supply is paid for at the marginal cost at the supply point. In the case where power lines form a loop and there is a limit on a line within the loop, the electrical requirements of providing supply force a reallocation of generation and a step change in the marginal costs. This is known as the spring washer effect. In some cases the cost of supply can reach extreme levels. Transpower and its customers are interested in obtaining a better understanding of these events.

Simplified equations determining power generation and pricing are derived. Solutions for the supply costs are obtained for several simple cases. Numerical examples and a graphical representation demonstrate the spring washer effect and how a spring washer responds to changes. Several possible changes that reduce the cost of supply are demonstrated.

When a spring washer comes active the marginal prices make a step change. This means that power costs can be be highly sensitive to the values of all the network parameters that affect the power flow in the power loop.

Detecting when a large spring washer is likely to occur is of considerable interest. Suggestions on how this could be done are made.

1. Introduction

Bulk electrical power is bought and sold at marginal prices, that is the price for all the power generated is set by the cost of providing ad-

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ditional power. Indicative power costs are calculated every five minutes, and final power prices are calculated on a half hour basis.

Power generation offers are made at a specified rate ($/MWh) for up to an upper limit (MW) of supply. A linear programming problem is solved to determine which power offers are accepted i.e dispatched. This is known as the scheduling, pricing and dispatch model or SPD. The power offers can be accepted for the full amount or a partial amount. The linear programming SPD chooses the lowest total of offer prices subject to constraints on the delivery of power. However the generated power is paid for at the marginal price at the generation nodes. For instance power offers of A: 100MW at $20/MWh and B: 200MW at $40/MWh would for a demand of 150MW, be dispatched at A: 100MW and B: 50MW and the total 150MW would be paid for at the marginal rate of $40/MWh. In this case only one power offer is partially dispatched.

The constraints ensure power can be delivered from the generators to the consumers. The electrical balance of the network provides some equality constraints, and limits on power line capacity provide inequality constraints.

A situation of particular interest is when the power lines form a loop. This then provides two paths from the generators to the power consumers i.e. the demands. If a limit in power line capacity is reached in one of the lines in the loop, power transmission needs to be balanced so that both the required power can be delivered and the limit is not exceeded. This typically requires the partial dispatch of two offers in a manner that satisfies both the electrical requirements of power delivery and the constraint. When this happens there is a step change in the marginal prices which then increase around the loop from one end of the line constraint to the other end. This is known as the spring washer.

Step changes in the marginal prices are applied to all the power used by a consumer, giving sudden changes in the total cost of power as the spring washer comes into effect. Under some circumstances the change in power prices around a spring washer becomes extreme. Transpower wanted information on when this can happen and how sensitive the resulting prices are to the electrical properties of the network.

The scheduling, pricing and dispatch (SPD) linear program that is used for calculation has some 600 nodes and 16000 constraints. In this report we describe a simplified form of the network equations in section 2 and the following section shows how the equations can be converted to a more convenient matrix form. The linear programming problem is described in section 4, with both the primal equations and the dual equations being derived. It is found that the linear programming equations have a relatively simple form with most elements being zero, and some analytic solutions are possible.

It is then shown in section 5 that in the absence of constraints only one generation offer is partially dispatched and the marginal costs throughout the network is equal to that of the partially dispatched generator. For the case where the only limiting power lines divide the power network into independent sections a similar solution can be found.

Section 7 examines the more interesting phenomena of the spring washer. A new graphical representation of the spring washer as a lever is given. This representation helps visualise the available data and thus get a better understanding of the spring washer. Several surprising properties of the spring washer are investigated, including reducing the cost to consumers by increasing the cost of generation.

The sensitivity of the marginal prices to the parameters of the network is discussed in section 8, and possible ways of determining if a serious spring washer can occur are discussed in the following section.

Symbols are defined as they are introduced and a table of symbols is given at the end of this report.

2. Basic equations

The generators to be used (generators dispatched) and the costs of supply are are determined by a large linear programming calculation which selects the lowest cost from the offers for supply and determines the marginal costs of supply at each node. Although it is convenient to use a matrix form for some calculations, the equations are initially developed for clarity in a scalar with subscripts form. Coding into a convenient vector and matrix format requires reordering of the subscripts.

The following subsections divide the equations into groups according to their origin. Then the next section presents the equivalent matrix equations.

2.1. Transmission equations

The transmission of power using an alternating current (AC) is described by an equation using complex numbers, that gives both the real power and the reactive power (as the complex part). The reactive power is due to electrical oscillations within the line and their associated losses. It must be supplied by the generators but is not available for use by the consumers. Typically the reactive power can be considered a constant fraction of the real power, that makes only a small adjustment to the calculations. As we are only interested in the real power the reactive power is not considered and a simplified equation written for the real power.

For alternating current (AC) in the line from node i to node j, the difference in phase angles $(q_i - q_j)$ determines the amount of power $(p_{i,j})$ transmitted along the power line (this is similar to voltage difference for a direct current line). The admittance of the line times the phase difference gives the power transmitted (it is however convenient for the later developments to use the reciprocal $b_{i,j}$ of the admittance in the equations) i.e.:

$$p_{i,j} = b_{i,j}^{-1}(q_i - q_j) \tag{1}$$

There is one degree of freedom in the q_i as they only occur as a difference. So one of the q_i can be set to zero, for instance the last value e.g.:

$$q_{n_{node}} = 0 \tag{2}$$

Alternatively q_1 can be set to a large value so that all the q_i are positive to simplify linear programming set up.

The second power equations ensure that power into each node (by generation or power lines) is equal to the power leaving the node (by demand or power lines) i.e. for the i^{th} node:

$$\sum_k g_{i,k} - d_i - \sum_j p_{i,j} = 0 \tag{3}$$

where d_i is the demand at node i and $g_{i,k}$ is the dispatched power generation for the k^{th} generation offer at node i. Remembering $p_{j,i} = -p_{i,j}$ summing equation 3 allows the $p_{i,j}$ terms to cancel leaving the power generated is equal to the total demand i.e.:

$$\sum_i \sum_k g_{i,k} - \sum_i d_i = 0 \tag{4}$$

2.2. Offers of supply

The owners of the generating capacity make offers for supply of certain amounts of power at an offer price. There can be multiple offers for supply at each node. Some offers may be high due to the cost of generation, or due to the need for the generating unit to default on a bulk supply contract and pay a penalty to be able to supply the power. With no loss of generality it is assumed that the offers for supply are in increasing order with the lowest prices first. The cost of supply at node i based on the offer prices is:

$$\sum_k s_{i,k} g_{i,k} \tag{5}$$

where $s_{i,k}$ is the price for the k^{th} offer at node i, and $g_{i,k}$ is the correspond amount of power dispatched. The total for all nodes:

$$c_{offer} = \sum_i \sum_k s_{i,k} g_{i,k} \tag{6}$$

This cost based on the offer prices is used in a linear programming formulation to determine which offers of generation capacity are accepted.

The total costs of transmission is:

$$c_{trans} = \sum_{p_{i,j}>0} (t_{i,j} abs(p_{i,j})) \tag{7}$$

where $p_{i,j}$ in the power flow in the line from node i to node j, and $t_{i,j}$ is the cost of transmission in \$/MWh. This can be added to the offer cost to give a more complete cost estimate for minimisation. However the major features of the SPD calculation can be demonstrated without the inclusion of this term.

However the generators are not paid according to the offers to supply, but at the marginal rate for that node (which is the highest offer accepted at the node with partially dispatched generation offers). The optimisation is used to select the generators dispatched and determine the marginal costs at each node.

2.3. Constraints

There are multiple constraints that apply to the scheduling pricing and dispatch calculation. We now examine these constraints.

The power dispatched from each offer is limited by the by the maximum available on that offer i.e.:

$$0 \leq g_{i,k} \leq g_{i,k}^{max} \tag{8}$$

The power in the lines is limited to a given maximum value but can be in either direction i.e.:

$$-p_{i,j}^{max} \leq p_{i,j} \leq p_{i,j}^{max} \tag{9}$$

Additional constraints come from security constraints that are applied to ensure that in the case of a line failure there is always time to correct line loads before line limits are exceeded (after a power increase there is a delay before the line temperature reaches the maximum allowable value). These have not been considered in the current analysis, but take the form:

$$u_{i,j,s,t} p_{i,j} + v_{i,j,s,t} p_{s,t} \leq w_{i,j,s,t} \tag{10}$$

Where $u_{i,j,s,t}, v_{i,j,s,t}$ & $w_{i,j,s,t}$ are constants determined by removing line s-t from the dispatch calculation.

3. Matrix notation

Having developed the required equations we now convert them to a more compact matrix format. However to get a convenient notation the scalars do not directly form the corresponding matrices. A new coding into vectors and matrices is needed as described below.

The values $g_{i,k}$ are placed in a column vector **g** in the order of the second subscript varying most rapidly (for instance 1,1; 1,2; 2,1; 3,1; 3,2; 3,3; . . .). Similarly $s_{i,k}$ and $g_{i,k}^{max}$ are placed in the column vectors **s** and $\mathbf{g}^{max}$. The total cost based on the offers (equation 6) is thus:

$$c_{offer} = \mathbf{s}'\mathbf{g} \tag{11}$$

The potentially non zero elements of $p_{i,j}$ in the forward direction (as per some arbitary convention) are placed in a column vector **p** in order of the second subscript varying most rapidly, and the elements q_i form the vector **q**.

A matrix **R** describes the connection of power lines to nodes. Each row of **R** corresponds to a power line in the same order as in **p**, while there is a column for each node in the network. A -1 in position ℓ, i indicates power line ℓ starts at node i, and a 1 in position ℓ, j indicates power line ℓ ends at node j. Zero values (the rest of the row) indicate the power line does not connect to the node. Thus a row of **R** corresponds to the power line in the same position in **p** with a one in the position corresponding to the node at the to end of the line, and minus one in the position corresponding to the node at the beginning of the line. Line power flow can be positive or negative, and the columns of **R** are not independent as they sum to zero.

Finally define **B** as the diagonal matrix with $b_{i,j}$ on the diagonal in the order corresponding to **p**.

Then equation 1 becomes:

$$\mathbf{Bp} = -\mathbf{Rq} \tag{12}$$

An equation such as 2 is still needed to resolve the degree of freedom in $\mathbf{q}$. Equation 2 becomes:

$$\mathbf{w}'\mathbf{q} = 0 \tag{13}$$

where $\mathbf{w}$ is a vector containing zeroes except for one in the last element.

The power balance at the nodes (equation 3) becomes:

$$\mathbf{F}\mathbf{g} + \mathbf{R}'\mathbf{p} = \mathbf{d} \tag{14}$$

Where $\mathbf{F}$ and $\mathbf{R}'$ have one row for each node. The i^{th} row of $\mathbf{F}$ contains ones in the position corresponding to the position of values $g_{i,k}$ in $\mathbf{g}$.

The constraints (equations 8 & 9) can also be written as matrix equations:

$$\mathbf{0} \le \mathbf{g} \le \mathbf{g}^{max} \ \ , \ \ -\mathbf{p}^{max} \le \mathbf{p} \le \mathbf{p}^{max} \tag{15}$$

4. The linear programming problem

Allocation the power dispatch is now a matter of solving the following linear programming problem:

Minimise (equation 11):

$$c_{offer} = \mathbf{s}'\mathbf{g} \tag{16}$$

with respect to $\mathbf{g}, \mathbf{p}$ and $\mathbf{q}$ and subject to the equality constraints (equations 12, 14 & 13) and inequality constraints (equations 15). Combining these equations gives:

$$\begin{bmatrix} \mathbf{0} & \mathbf{B} & \mathbf{R} \\ \mathbf{F} & \mathbf{R}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{w}' \end{bmatrix} \begin{bmatrix} \mathbf{g} \\ \mathbf{p} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{d} \\ 0 \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{g} \\ \mathbf{p} \\ \mathbf{q} \end{bmatrix} \ge \begin{bmatrix} -\mathbf{g}^{max} \\ \mathbf{0} \\ -\mathbf{p}^{max} \\ -\mathbf{p}^{max} \end{bmatrix} \tag{18}$$

The variables other than $\mathbf{g}, \mathbf{p}$ & $\mathbf{q}$ used in these equation to form the vectors and matrices (i.e. for nodes: d_i, lines: $b_{i,j}$, $p_{i,j}^{max}$, offers: $s_{i,k}$, $g_{i,k}^{max}$) have values that are supplied as data to the linear programming problem.

The marginal prices to the consumer are then obtained as the Lagrange multiplier values, which are the solution to the dual linear programming problem. The marginal prices are the derivative of the cost being minimised with respect to the constant terms in the constraints. For the problem: minimise cost $\mathbf{c}'\mathbf{x}$ wrt $\mathbf{x}$ subject to the active constraints $\mathbf{Ax} = \mathbf{b}$, the Lagrange multipliers $\boldsymbol{\lambda}$ satisfy $\boldsymbol{\lambda}'\mathbf{A} = \mathbf{c}'$. Now the derivative of the cost is:

$$\frac{\partial \mathbf{c}'\mathbf{x}}{\partial b_i} = \frac{\partial \mathbf{c}'\mathbf{A}^{-1}\mathbf{b}}{\partial b_i} = \frac{\partial \boldsymbol{\lambda}'\mathbf{b}}{\partial b_i} = \lambda_i \tag{19}$$

Most linear programming programs will calculate these marginal costs.

Introducing the Lagrange multipliers (as Greek letters) for the constraints in the power problem gives the conditions:

$$\mathbf{diag}\left(\begin{bmatrix} \boldsymbol{\mu} \\ \boldsymbol{\nu} \\ \boldsymbol{\zeta} \\ \boldsymbol{\xi} \end{bmatrix}\right)\left(\begin{bmatrix} -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{g} \\ \mathbf{p} \\ \mathbf{q} \end{bmatrix} - \begin{bmatrix} -\mathbf{g}^{max} \\ \mathbf{0} \\ -\mathbf{p}^{max} \\ -\mathbf{p}^{max} \end{bmatrix}\right) = 0 \tag{20}$$

where $\boldsymbol{\mu}$ corresponds to power offers at the upper limit, $\boldsymbol{\nu}$ to power offers at zero, $\boldsymbol{\zeta}$ to power lines at the upper limit, and $\boldsymbol{\xi}$ to power lines at the lower limit. Here if the constraint is active its value is zero and the corresponding Lagrange element is positive, while inactive constraints have a positive value and a zero Lagrange element. The non zero Lagrange multiplier elements are calculated from the following equation (note ψ is a scalar while the other Lagrange multipliers are vectors):

$$\begin{bmatrix} \mathbf{0} & \mathbf{F}' & \mathbf{0} & -\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{B} & \mathbf{R} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{I} \\ \mathbf{R}' & \mathbf{0} & \mathbf{w} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \boldsymbol{\tau} \\ \boldsymbol{\delta} \\ \psi \\ \boldsymbol{\mu} \\ \boldsymbol{\nu} \\ \boldsymbol{\zeta} \\ \boldsymbol{\xi} \end{bmatrix} = \begin{bmatrix} \mathbf{s} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{21}$$

where $\boldsymbol{\tau}$ corresponds to the power lines, $\boldsymbol{\delta}$ corresponds to the nodes, and ψ to the additional condition on $\mathbf{q}$ (equation 13. It is the $\boldsymbol{\delta}$ that are of particular interest as they are the marginal power costs at the nodes used for both paying the generators and for charging the consumers.

The decisions on which elements of the Lagrange multipliers are zero are made using a linear programming algorithm. The number of zero elements will be such that there are $n_{trans} + n_{node} + n_{offer}$ non zero elements matching the number of equations for the Lagrange multipliers.

4.1. About the LP equations

The dispatch of power offers at the nodes will always be such that at most one offer at each node is not at a limit, and typically most nodes will have either no power offers or all power offers at a limit. Further at each node all the lower cost offers will be accepted and the higher cost offers rejected. This considerably reduces the options that need to be considered during the linear programming.

There are $n_{node}+n_{trans}+1$ equality equations (17). This is sufficient to determine the transmission powers $\mathbf{p}$ (n_{trans} elements) and their driving phase angles $\mathbf{q}$ (n_{node} elements) plus one value of the generation allocations $\mathbf{g}$. Each active constraint on an element of the power transmission flows $\mathbf{p}$ requires one more of the $\mathbf{g}$ to be moved off a constraint (15, 18) and instead be determined by equations 17.

At each node the constraints on the offer power amounts $\mathbf{g}$ are such that at most one amount can be not be determined by a constraint (15, 18). Once these constraints are determined, the constrained power offers are fixed amounts of power that can be combined with the demand $\mathbf{d}$ for the solution of the remaining equations.

Once the dispatch has been determined by the LP algorithm, the offers constrained at the upper limit can (as they are constant) be combined with the demand terms, and those at zero can be removed from the calculation. It is then convenient to allocate one offer, that is either zero or unconstrained, per node (most will be zero), then $\mathbf{F}$ reduces to the unit matrix. This also sets $\boldsymbol{\mu}$ to zero, and the values of $\boldsymbol{\nu}$ corresponding to the unconstrained power offers to zeros. Now the top partition of equation 21 gives:

$$\delta_k = s_k \tag{22}$$

for the unconstrained offers (where $\nu_k = 0$) thus giving the marginal cost at the generating nodes equal to the generation cost as expected. The remaining equations in the top partition, as ν does not occur in other equations, determine the remaining values of ν. Thus for both constrained and unconstrained offers:

$$\nu_k = s_k - \delta_k \tag{23}$$

The second partition row of equation 21 gives, for an unconstrained power line from node i to node j, the difference in the marginal costs at the nodes:

$$\delta_j - \delta_i = b_{i,j}\tau_{i,j} \tag{24}$$

For a constrained power line from node I to node J, we have either $\zeta_{I,J} > 0$ or $\xi_{I,J} > 0$, assume it is $\zeta_{I,J} > 0$. This gives one more unknown in equation 21 which is allowed by forcing an additional value of $\boldsymbol{\nu}$ to zero. In the primal the corresponding value (to the zero $\boldsymbol{\nu}$ element) in $\mathbf{g}$ has to be calculated and $p_{I,J}$ is fixed at its maximum.

The third partition row of equation 21 gives:

$$\psi = 0 \tag{25}$$

as the rows of $\mathbf{R}$ sum to zero. This corresponds to the equation 2 that removes the redundancy in $\mathbf{q}$ and thus has no effect on the value being minimised c_{offer}. Further for any node that has only one incoming power line $L1$ and one out going power line $L2$:

$$\tau_{L1} = \tau_{L2} \tag{26}$$

hence the values of τ divide into blocks of equal values for section of the network where the lines and nodes are in a simple series connection.

The equality constraints which are the power distribution equations 17 can be written, after removing the dependent column of $\mathbf{R}$ (giving $\mathbf{R}^{\#}$) and corresponding rows of $\mathbf{d}$ and $\mathbf{F}$ (giving $\mathbf{d}^{@}$ & $\mathbf{F}^{@}$ as:

$$\begin{bmatrix} \mathbf{B} & \mathbf{R}^{\#} & \mathbf{0} \\ (\mathbf{R}^{\#})' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{q}^{@} \\ q_{n_{node}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{d}^{@} \\ 0 \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{F}^{@} \\ \mathbf{0} \end{bmatrix} \mathbf{g} \tag{27}$$

The matrix on the left hand side is now invertible and hence these equations can be solved for $\mathbf{p}$ and $\mathbf{q}$ giving:

$$\begin{bmatrix} \mathbf{p} \\ \mathbf{q}^{@} \\ q_{n_{node}} \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \mathbf{R}^{\#} & \mathbf{0} \\ (\mathbf{R}^{\#})' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathbf{0} \\ \mathbf{d}^{@} \\ 0 \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{F}^{@} \\ \mathbf{0} \end{bmatrix} \mathbf{g} \right) \tag{28}$$

The coefficient matrix can be factorised as follows, and thus the solution to equation 27 or the inverse in equation 28 can be found:

$$\begin{bmatrix} \mathbf{B} & \mathbf{R}^{\#} \\ (\mathbf{R}^{\#})' & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ (\mathbf{R}^{\#})' & \mathbf{L} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{B}^{-1}\mathbf{R}^{\#} \\ \mathbf{0} & -\mathbf{L}' \end{bmatrix} \tag{29}$$

where:

$$\mathbf{L}\mathbf{L}' = (\mathbf{R}^{\#})'\mathbf{B}^{-1}\mathbf{R}^{\#} \tag{30}$$

and $\mathbf{L}$ is the lower triangular Cholesky factor. As $\mathbf{R}$ is sparse calculating $\mathbf{L}$ is relatively easy. Note that if p_I is determined by a constraint the corresponding row of equation 28 becomes a constraint on the values on $\mathbf{g}$.

5. Case of no power transmission constraints

If none of the constraints on power lines are active, equation 28 is satisfied by $\mathbf{p}$ & $\mathbf{q}$. This leaves one equality to be satisfied. Summing the rows of equation 14 gives equation 4 which can be rewritten as:

$$\mathbf{u}'\mathbf{g} = \mathbf{u}'\mathbf{d} \tag{31}$$

where $\mathbf{u}$ is a vector of ones with compatible size. This equation needs to be satisfied while minimising equation 16. The lowest cost sources are dispatched in order until equation 31 can be satisfied. One supply offer, a $g_{i,k}$ (the one with the highest accepted offer price) will be used to satisfy equation 31. Offers with a lower supply offer prices will have $g_{i,k}$ at its maximum, and those with higher supply prices will have $g_{i,k}$ at zero.

For notational convenience combine all the offers accepted at maximum power with $\mathbf{d}$, and assign one offer to each node, one of which $g_{I.1}$ is not zero and corresponds to the supply offer partially accepted as described above. The offers at other nodes will have offer prices greater than $g_{I,1}$ and $g_{i,1} = 0$. This makes $\mathbf{F} = \mathbf{I}$. The equations for the optimum values are then:

$$\begin{bmatrix} \mathbf{0} & \mathbf{B} & \mathbf{R} \\ \mathbf{F} & \mathbf{R}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{w}' \\ \mathbf{I} - [1_{I,I}] & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{g} \\ \mathbf{p} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{d} \\ 0 \\ \mathbf{0} \end{bmatrix} \tag{32}$$

where $[1_{I,I}]$ is a matrix with one in the I, I position and zeros elsewhere. The I^{th} row of the third partition is all zeros making the number of equations equal to the number of variables, and hence the equations can be solved.

The Lagrange multipliers $\boldsymbol{\mu}$, $\boldsymbol{\zeta}$ and $\boldsymbol{\xi}$ are zero as the corresponding constraints are not active. The equation (from 21 and 32) for the remaining Lagrange multipliers is:

$$\begin{bmatrix} \mathbf{0} & \mathbf{F}' & \mathbf{0} & \mathbf{I} - [1_{I,I}] \\ \mathbf{B} & \mathbf{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{R}' & \mathbf{0} & \mathbf{w} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\tau} \\ \boldsymbol{\delta} \\ \psi \\ \boldsymbol{\nu} \end{bmatrix} = \begin{bmatrix} \mathbf{s} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{33}$$

One column corresponding to zero valued ν_I is all zeros. The solution to these equations is:

$$\boldsymbol{\tau} = 0 \ , \ \ \boldsymbol{\delta} = s_I \ , \ \ \psi = 0 \ , \ \ \boldsymbol{\nu} = \mathbf{s} - s_I \tag{34}$$

Hence in the case of no constraints on power transmission, the marginal node prices $\boldsymbol{\delta}$ are equal to the highest accepted offer price.

6. Case of a limiting power line dividing the network

If the only power line linking two parts of the power network becomes limiting, the network is effectively divided into two parts, with the limiting line acting as a fixed demand in one part and a dispatched offer in the other.

The phase angle values $\mathbf{q}$ have a degree of freedom of each side of the of the constraint. On one side this degree of freedom can be handled, as before by setting a chosen value to zero. Then on the other side the degree of freedom is used to get the required power flow in the line linking the two sides. The marginal cost of power will be constant in each section that has no constraints, but different between sections.

This analysis can easily be extended to multiple single lines that divide the network into multiple sections.

7. The spring washer effect

A spring washer occurs when the the power lines form a loop and there is an active limit on the power flow in one of the power lines. When this limit becomes active there is a sudden change in the electrical requirements within the loop. The phase angle at the end of the constrained power line must satisfy the requirements of the constrained line and also the requirements of power flow around the remainder of the loop.

The simplest case of a spring washer, which is the one examined here is where the network is only a closed loop. A closed loop without a constraint on power flows is the same as treated in section 5.

If a constraint occurs on one of the power lines in a loop this extra constraint requires another constraint to cease to be active so that a variable is freed to allow the constraint on power transmission to be satisfied. This is done by moving one more power offer from its constraint. So for a loop with one power line constraint, two generation amounts need to be unconstrained in equations 9 (which is part of equation 18) so that these values can be used to satisfy the equality constraints 17.

For a simple loop the network matrix $\mathbf{R}$ is square and both rows and columns contain only one value of 1 and one value of -1. As above we allow only one generation offer at each node after combining fully dispatched offers with the demand and removing those that will never be dispatched, so the $\mathbf{F} = \mathbf{I}$. Let I and J be the two nodes where power dispatch needs to be calculated, and L be the line with the constraint, then equations for the Langrange multipliers are:

$$\begin{bmatrix} \mathbf{0} & \mathbf{F}' & \mathbf{0} & \mathbf{I} - [1_{I,I}] - [1_{J,J}] & \mathbf{0} \\ \mathbf{B} & \mathbf{R} & \mathbf{0} & \mathbf{0} & -[1_{L,L}] \\ \mathbf{R}' & \mathbf{0} & \mathbf{w} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\tau} \\ \boldsymbol{\delta} \\ \psi \\ \boldsymbol{\nu} \\ \boldsymbol{\xi} \end{bmatrix} = \begin{bmatrix} \mathbf{s} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{35}$$

As the rows of $\mathbf{R}'$ add to zero ψ is equal to zero (equation 25).

As the rows of $\mathbf{R}'$ contain only a 1 and -1, $\boldsymbol{\tau}$ is constant (value τ_0), the changes in between adjacent values of $\boldsymbol{\delta}$ is proportional to values of $\mathbf{b}$ for the line between the nodes. There are two values of $\boldsymbol{\nu}$ that are zero.

If we number the nodes starting at the low cost end of the line power constraint (in the constrained line power flows from the low cost end to the high cost end) going around the loop toward the high cost end of the constrained line, δ increases by $b_{i,i+1}\tau_0$ at each step from node i to node i+1, and thus:

$$\delta_i = \delta_1 + \tau_0 \sum_{j=1}^{i-1} b_{j,j+1} \tag{36}$$

This contains two constants δ_1 & τ_0 that are determined from the two accepted power offers (p_I at cost s_I and p_J at cost s_J with $I < J$) in the loop. These set $\delta_I = s_I$ and $\delta_J = s_J$. Hence:

$$\tau_0 = \frac{s_J - s_I}{\sum_{j=I}^{J-1} b_{j,j+1}} \tag{37}$$

and:

$$\delta_1 = s_I - \tau \sum_{j=1}^{I-1} b_{j,j+1} \tag{38}$$

A convenient way to examine this result is a plot of the marginal costs δ_i verses the sum $\sum_{j=1}^{i-1} b_{j,j+1}$ (see figure 1). This is a straight line with its position determined by the two generators that have their power offer partly accepted. This plot makes it clear how the marginal prices are controlled by the offer prices and the line properties. A large price difference between the generators, or a high admittance (low $b_{i,j}$) line between the generators gives a steep slope to the lever line. Then large

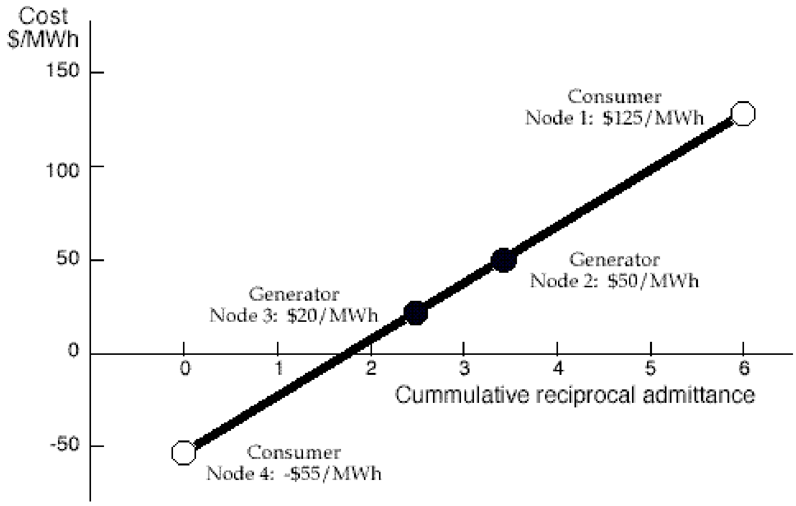

Figure 1. Representation of a spring washer as a lever

marginal prices are obtained when there is node distant from the generators on the high side of the lever. Similarly on the low side negative power prices may be seen.

The property of particular interest regarding the spring washer is the sensitivity of the conditions under which it comes into effect and the size of the change as it comes into effect. The following subsections use a simple loop network to demonstrate some properties of the spring washer.

7.1. An example of the spring washer

The network used to demonstrate a spring washer is shown in figure 2. Four nodes connected in loop, with a demand of 400 MW at node 1 and a limit of 200 MW on the line from node 4 to node 1. A generator at node 2 offers power of up to 1000MW at $50/MWh, and at node 3 offers of 200MW at $10/MWh and up to 1000MW at $20/MWh are made.

Table 1 shows the result of the SPD linear program calculation for this configuration. Power of 400MW is dispatched from node 3 (see column Actual under Generation for the two values adding to 400), and

Table 1. Base case network dispatch and marginal pricing

```
LP cost =    6000.0
Generation payment =    8000.0
Demand payment     =    8000.0
```

Node	Demand	$/MWh	Generation Max	Cost	Actual	w
1	400.0	20.00				
2	0.0	20.00				
			1000.0	50	0.0	0.0
3	0.0	20.00				
			200.0	10	200.0	10.0
			1000.0	20	200.0	0.0
4	0.0	20.00				

Lines Start	End	Power transmission Max	Admit.	Cost	Actual	w
1	2	500.0	0.40	0	-200.0	0.0
2	3	500.0	1.00	0	-200.0	0.0
3	4	500.0	0.40	0	200.0	0.0
4	1	200.0	1.00	0	200.0	0.0

this is paid for at $20/MWh which is the marginal price at that node ($/MWh column). Note that the generation payment of $8000 is higher than the sum of the offer prices of $6000 which is used in the linear programming optimisation. The marginal cost of power at all of the nodes is $20/MWh. Through the network 200MW flows in both directions around the loop from node 3 to node 1. This is right on the upper limit on capacity of the line from node 4 to node 1, however this constraint is not active in the LP solution as can be seen by the corresponding Lagrange multiplier w.

As this example is right on the limiting capacity of the line from node 4 to 1, a minor change to the demand or line admittances can make the constraint active. Table 2 shows the effect of increasing the demand at mode 1 by 0.1MW. This brings the spring washer into effect with the constraint on line from node 4 to node 1 becoming active. Due to the electrical requirements of the loop to deliver power it is necessary to reduce generation at node 3, and generate at a higher cost at node 2.

Table 2. Effect of a minor increase in demand - spring washer active

```
LP cost =    6012.5
Generation payment =    8012.5
Demand payment     =   50012.5
```

Node	Demand	$/MWh	Generation Max	Cost	Actual	w
1	400.1	125.00				
2	0.0	50.00				
			1000.0	50	0.35	0.0
3	0.0	20.00				
			200.0	10	200.00	10.0
			1000.0	20	199.75	0.0
4	0.0	-55.00				

Lines Start	End	Power transmission Max	Admit.	Cost	Actual	w
1	2	500.0	0.40	0	-200.10	0.0
2	3	500.0	1.00	0	-199.75	0.0
3	4	500.0	0.40	0	200.00	0.0
4	1	200.0	1.00	0	200.00	210.0

The price difference between the two generators controls the slope of the line giving the marginal costs as seen in the $/MWh column of table 2. There is an instantaneous jump from constant marginal costs (of \$20/MWh in table 1) to the varying costs given by equation 36 (-\$55/MWh to \$125/MWh in table 2). When this jump occurs is determined by the power demand which sets the line power, and by the limit of the line capacity. The line admittances ($1/b_{i,j}$) determine the marginal prices after the spring washer becomes active, The marginal prices then vary in a smooth manner with the line properties ($b_{i,j}$) as shown in equations 36, 37, & 38. Comparing table 1 with table 2 we see the LP cost (based on offer prices) and the Generation payment have risen slightly due to a reduction in the generation at node 3 and an increase at the more expensive node 2 (cost rise is $0.35 * 50 - 0.25 * 20 = 12.5$), however there is a jump of \$42000 plus the cost of the extra 0.1 MW of demand at \$125/MWh in the "Demand payment". This jump of \$42000 is due to the increase in demand at node 1 (of 0.1MW), requiring

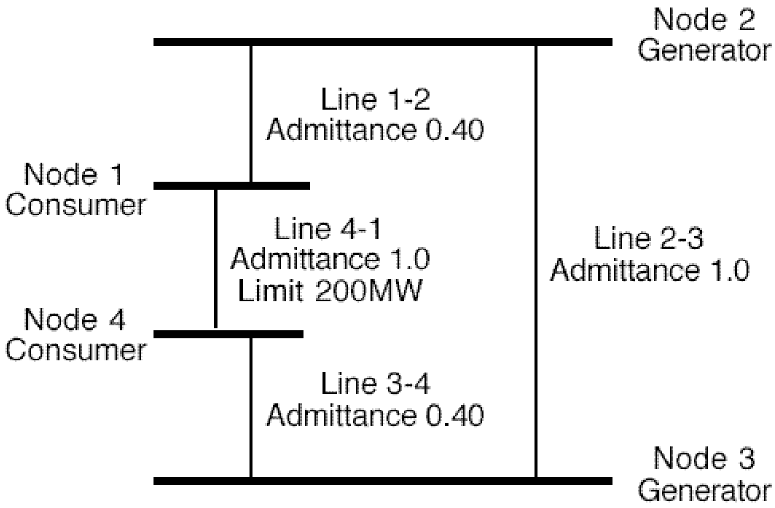

Figure 2. Power network used in the numeric examples

the generator at node 2 to be introduced at a higher marginal cost, which is then applied to all the power supplied at node 1. The consumers at node 1 may well be upset by this.

Large changes in marginal supply costs can come from a situation where the generators are (electrically) close together, are offering different prices, and the demand is relatively distant. This can be envisaged as a long lever (of the type in figure 1) with the controlling points close together.

7.2. Behaviour of the spring washer

A spring washer occurs due to constraints on power delivery and the cost structure of generation. Changes to these can have a considerable and somewhat surprising effects on the spring washer.

The marginal costs at the nodes, for a given dispatch of power generation, depend on the generation costs. Increasing the marginal cost of generation (table 3) at node 3 reduces the effect of the spring washer until the value of \$50/MWh is reached, when all nodes have equal (at \$50/MWh) marginal rates. If the offer at node 3 goes above \$50/MWh the linear programming allocates generator 2 as the lowest cost genera-

Table 3. Effect of increasing the offer price at node 3 - spring washer active

LP cost = 12005.0
Generation payment = 20005.0
Demand payment = 20005.0

Node	Demand	$/MWh	Generation Max	Cost	Actual	w
1	400.1	50.00				
2	0.0	50.00				
			1000.0	50	0.3	0.0
3	0.0	50.00				
			200.0	10	200.0	40.0
			1000.0	50	199.7	0.0
4	0.0	50.00				

Lines Start	End	Power transmission Max	Admit.	Cost	Actual	w
1	2	500.0	0.40	0	-200.1	0.0
2	3	500.0	1.00	0	-199.7	0.0
3	4	500.0	0.40	0	200.0	0.0
4	1	200.0	1.00	0	200.0	0.0

tion (as in table 4). In this case the transition to a spring washer (table 4 to table 3) does not create a step change as the spring washer comes into effect with equal generation costs. However if the generation dispatch of table 2 is maintained the cost to the users (Demand payment) will continue to decrease as the offer price at node 3 increases above $50/MWh. The marginal costs are a linear function of the offer prices, as can be seen in equation 21. One side of the limiting power line the costs will decrease while on the other side the costs increase as the generation prices are changed.

Table 4 shows the alternative power dispatch for the conditions of tables 3 & 2. In table 4 the power dispatch is no longer held at the lower cost node 2 but has moved to node 3 with the result that the line 1 to 4 is no longer limiting and the spring washer is no longer present. This was done by increasing the cost (by a small amount not vissible in table 4) of the generation offer at node 3, so that the alternative dispatch at node 2 is accepted by the linear programming algorithm. The same result can be obtained by removing generation at node 3.

Table 4. Effect of increasing the offer price at node 3 - spring washer inactive

```
LP cost =   12005.0
Generation payment =   20005.0
Demand payment     =   20005.0
```

Node	Demand	$/MWh	Generation Max	Cost	Actual	w
1	400.1	50.00				
2	0.0	50.00				
			1000.0	50	200.1	0.0
3	0.0	50.00				
			200.0	10	200.0	40.0
			1000.0	50	0.0	0.0
4	0.0	50.00				

Lines Start	End	Power transmission Max	Admit.	Cost	Actual	w
1	2	500.0	0.40	0	-228.6	0.0
2	3	500.0	1.00	0	-28.5	0.0
3	4	500.0	0.40	0	171.5	0.0
4	1	200.0	1.00	0	171.5	0.0

If the power limit on the line from node 4 to node 1 is reduced to 100MW the dispatch of generation becomes infeasible. The generation required at node 3 needs to be negative (-300MW in fact) to satisfy the power equations. The addition of a demand at node 3 as shown in table 5 allows the problem to become feasible.

Table 6 shows that a lower demand at node 4 can have the same effect of making the dispatch of generators feasible.

Removing the constrained power line reduces the node cost back to $20/MWh at all nodes (table 7). However the power flow in some of the remaining lines increases, and it is possible that this violates the constraints on those power lines making this case infeasible. If the power delivery without the limiting power line is feasible, the consumer at node 1 is much better off with this option than that of table 2.

Table 5. Increasing power consumption to make feasible - spring washer active

```
LP cost =   36000.0
Generation payment =   36000.0
Demand payment     =   64000.0

Node  Demand     $/MWh          Generation
                             Max   Cost  Actual    w
   1   400.0    150.00
   2     0.0     50.00
                            1000.0    50  700.0    0.0
   3   400.0     10.00
                             200.0    10  100.0    0.0
                            1000.0    20    0.0    0.0
   4     0.0    -90.00

Lines          Power transmission
Start  End    Max   Admit.  Cost  Actual     w
   1    2    500.0   0.40     0   -300.0     0.0
   2    3    500.0   1.00     0    400.0     0.0
   3    4    500.0   0.40     0    100.0     0.0
   4    1    100.0   1.00     0    100.0   280.0
```

7.3. Interconnected spring washers

Certain connections between spring washers have the potential to provide very large changes in marginal prices at certain nodes. Large step changes in marginal costs of power can occur if one spring washer supplies power to a second spring washer. Power suppled at the marginal rate at the high cost end of the first spring washer can replace one generator in the second spring washer. A large change in the cost of power entering the second spring washer creates larger changes in marginal rates within the second spring washer.

Table 8 and figure 3 consist of two spring washers with each being similar to the previous tables. The high cost side (node 1) of a spring washer (nodes 1, 2, 3 & 4; with a limit on the line from node 1 to 4) is the source of power to (node 7) a second similar spring washer (nodes 5, 6, 7 & 8; with a limit on the line from node 5 to 8). The first spring washer behaves as in the previous examples. The second spring washer requires

Table 6. Smaller increase of power consumption at node 4 to make feasible - spring washer active

```
LP cost =   23900.0
Generation payment =   23900.0
Demand payment     =   51900.0

Node  Demand     $/MWh          Generation
                            Max   Cost  Actual    w
   1   400.0    150.00
   2     0.0     50.00
                          1000.0    50  475.0    0.0
   3     0.0     10.00
                           200.0    10   15.0    0.0
                          1000.0    20    0.0    0.0
   4    90.0    -90.00

Lines          Power transmission
Start  End    Max   Admit.  Cost  Actual      w
   1    2    500.0   0.40     0   -300.0     0.0
   2    3    500.0   1.00     0    175.0     0.0
   3    4    500.0   0.40     0    190.0     0.0
   4    1    100.0   1.00     0    100.0   280.0
```

power at node 7 to enable power delivery. This is being supplied from node 1 where the marginal cost is \$125, some what higher than the cost of \$50 at the corresponding node 2 of the first spring washer. The result is a higher slope in the second spring washer and thus larger changes in marginal costs giving a cost of \$387.5 at node 8. This is now more than three times higher than the corresponding node (node 1) in the first spring washer. Changes to parameters in the network can greatly increase the size of this effect.

This particular configuration and parameter values is close to being infeasible and some small changes have been found to make the associated linear program infeasible. As with the simple spring washer increasing the power costs at nodes 3 and 6 to \$50 levels the spring washers giving a marginal price at all nodes of \$50/MWh.

Table 7. Effect of removing the constrained line

```
LP cost =    6000.0
Generation payment =    8000.0
Demand payment     =    8000.0

Node  Demand     $/MWh          Generation
                           Max   Cost  Actual    w
   1   400.0     20.00
   2     0.0     20.00
                         1000.0    50     0.0    0.0
   3     0.0     20.00
                          200.0    10  200.0   10.0
                         1000.0    20  200.0    0.0
   4     0.0     20.00

Lines          Power transmission
Start  End    Max   Admit.  Cost  Actual     w
   1    2   500.0   0.40     0   -400.0     0.0
   2    3   500.0   1.00     0   -400.0     0.0
   3    4   500.0   0.40     0     -0.0     0.0
```

8. Effect of network parameters

The network properties are described by the line admittances and power line capacities. These are not exact values and Transpower is interested in how changes in these values can effect the marginal prices.

As we have seen above very small changes in the network values, when a spring washer becomes active, do create large changes in the marginal costs. This is a step change caused by the linear programming solution changing from one set of constraints to another slightly different set of constraints. It needs only a minimal change in in one of the network values to move from one side of the constraint to the other. As the marginal costs are used for charging, the effect of such a step change on consumers can be large as is seen in the change from table 1 to table 2.

If there are no constraints active the network is not sensitive to the data values. When a spring washer is active, figure 1 indicates when small changes can have a magnified effect. In particular if the level arm is long and the distance between the generators is small a change in

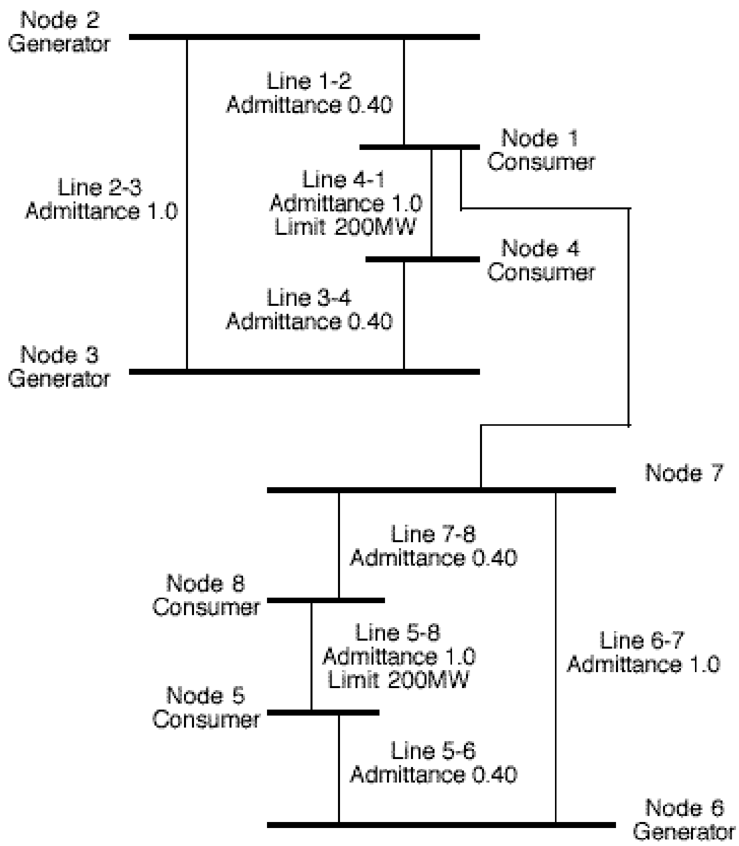

Figure 3. Power network for interconnected spring washers

the line admittance between the generators can change the line slope significantly, resulting in a large change at the end of the lever.

9. Locating potential spring washers

It is of considerable to Transpower and its customers to determine when a large spring washer can occur. Several suggestions have been made on how a potential spring washer can be detected. The following subsections explain the possibilities proposed.

9.1. Investigation of near optimal linear programming vertices

The sensitivity of the linear programming solution can be investigated by examining nodes of the simplex adjacent to the solution. This done by manipulating the linear programming matrix to exchange an active constraint with an inactive constraint. The exact calculation depends on the matrix form used by the linear programming program and is often an option in linear programming computer programs.

As the scheduling, pricing and dispatch problem is in a high dimensional space it is not obvious how many adjacent nodes of the simplex need to be examined. Only those nodes that are associated with large changes to the Lagrange multipliers need to be investigated. It is not clear how many alternatives need to be examined. This needs to be investigated.

A similar approach is to examine linear programming solutions that are within a certain percentage of the optimum.

9.2. Random perturbation of linear programming data

Another possibility is to use random perturbations to the demand (or other network parameters) to determine the likelihood of a serious spring washer occurring. A known seed for the pseudo random number generation would be needed as there is a need for all calculations to be repeatable for verification purposes.

Records of typical variation in demand can be used to determine the size of the random variation used.

9.3. Investigation of constraints close to becoming active

As a spring washer can only occur where the power lines form a loop and a constraint on power occurs, a first step is to locate lines within a loop that are close to their limit. Reducing the limit on these lines so that a spring washer occurs can be used to determine how serious the effect of the spring washer will be. Similar to that suggested above, records of the amount of variation of power in the lines can be used to determine which lines are close to their limit.

Depending on the number of lines close to their limits, testing one line at a time could be tedious, and certainly there could be a large number of combinations lines to test. Random variation of the line limits could be used to test for possible large spring washers. Again the amount of variation used can be related to the typical variation seen in the line power over time periods similar to those of the current situation.

10. Conclusions and recommendations

The simplified formulation presented is considered to include the main features of the scheduling, pricing and dispatch (SPD) model used by Transpower. Power line costs are expected to make only relatively small changes to the cost structure. The security constraints (section 2.3), having two power lines in the constraint equation offer many different ways constraints can occur within the power network. These warrant further investigation. It is expected that if there is only limited interaction between the two lines in the constraint the situation will be similar to a simple constraint. On the other hand if the two lines do interact as when they are both in the same power line loop, a larger effect is to be expected.

The linear programming equations have been derived based on the simplified formulation, and found to take a relatively simple form. For certain cases an analytic solution for the marginal costs has been found.

When no constraints are active the solution becomes simple with one generator balancing generated and demand powers, and a constant marginal price throughout the network. Constraints that divide the network into sections give separate solutions for each section that are similar to that for no constraints.

When power lines form a loop a spring washer occurs when one line reaches a constraint. When this happens there is a step change in the marginal costs and there is a systematic increase in marginal supply costs around the spring washer loop. As power is charged at marginal prices the additional cost, due to this step change, to a consumer can be large. A method of plotting the spring washer as a lever has been developed, and this provides a convenient way of understanding the behaviour of the spring washer.

Increasing the supply cost on the lower power supply, levels the spring washer and can significantly reduce the costs to a consumer at the high end of the spring washer. At the same time the marginal costs at the low end of the spring washer increase. Further increases in the price without redispatching the supply, can reduce these customer rates to zero or even negative values.

In some cases the spring washer can vanish if the requirement of minimal cost of supply is removed so that more costly generation can be used, but in spite of this increase, the resulting consumer price decreases. However node costs on the low side of the spring washer increase.

Simply removing the limiting line from the network removes the spring washer and returns the costs to the values before the spring washer came into effect. However this could create unacceptable loads in the remainder of the network.

Step changes in marginal costs occur when a spring washer forms requiring a change in dispatch. These are true steps that will occur with a very small change in parameter values. Thus there is always an extreme sensitivity to the network parameters as the spring washer forms.

In normal circumstances, other than the step change as spring washer forms, the network performance is a continuous function of the parameters. The lever representation of the spring washer indicates the conditions under which the marginal costs are more sensitive to the line admittances.

Detecting a potential spring washer is needed by Transpower and their customers. Three methods have been suggested. The third is probably the most practical, and involves identifying constraints that are currently close to limits and then adjusting the limit in the scheduling, pricing and dispatch calculation to create the spring washer so that the magnitude of its effect can be determined.

The supply, pricing and dispatch (SPD) model used minimises the cost of the generation offers. This has been seen to generate some undesirable effects. Minimising the total cost of supply based on the marginal cost to consumers, subject to this being greater than the total cost of generation (and transmission) based on its marginal costs, may be a more rational approach to the SPD calculations.

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Symbols

The symbols used are described in the following sections according to the type of symbol.

Subscripts

i	Node
I	A fixed node
j	Node
J	A fixed node
k	Power offer (upper limit depends on associated node)
ℓ	Index for power line from node i to node j when power lines are ordered with the j subscript varying most rapidly
L	A fixed value of the power line index ℓ
$L1, L2$	Fixed values of ℓ
r	Index for k^{th} offer for generation at node i when all the offers are ordered with the k subscript varying most rapidly
s	Node
t	Node

Dimensions

n_{offer}	Total number of offers for power supply
n_{gen}	Number of nodes with a generator not at a limit
n_{node}	Number of nodes
n_{trans}	Number of power transmission lines

Symbols related to a node

d_i	Demand at node i (MW)
q_i	Voltage phase angle at node i
$\mathbf{d}$	Column vector containing d_i
$\mathbf{q}$	Column vector containing q_i
$\boldsymbol{\delta}$	Column vector length n_{node} of Lagrange multipliers for node equations. These are the marginal power costs at the nodes
ψ	Langrange multiplier for $q_{n_{node}} = 0$

Symbols related to the power offers

$g_{i,k}$	Power dispatched at node i from the k^{th} power offer note there can be zero or more offers at each node
$g_{i,k}^{max}$	Maximum amount (MW) at node i of k^{th} power offer
$s_{i,k}$	Price ($/MWh) at node i for the k^{th} power offer
$\mathbf{g}$	Column vector length n_{offer} containing $g_{i,k}$ in the order of k varying most rapidly
$\mathbf{g}^{max}$	Column vector length n_{offer} containing $g_{i,k}^{max}$ (see $\mathbf{g}$)
$\mathbf{s}$	Column vector length n_{offer} containing $s_{i,k}$ (see $\mathbf{g}$)
$\boldsymbol{\mu}$	Column vector of length n_{offer} of Lagrange multipliers for offer upper limits of $\mathbf{g}^{max}$
$\boldsymbol{\nu}$	Column vector of length n_{offer} of Lagrange multipliers for offer lower limits of zero

Symbols related to power lines; note that these are non zero only if a power line connects node i to node j:

$b_{i,j}$	Reciprocal of the admittance of power line from node i to node j. For convenience in the equations we have used the reciprocal.
$p_{i,j}$	Actual power transmission in power line from node i to node j note $p_{j,i} = -p_{i,j}$
$p_{i,j}^{max}$	Maximum power capacity (Mw) of power line from node i to j
$t_{i,j}$	Cost (\$/Mwh) of transmission from node i to j
$\mathbf{B}$	Square matrix of size n_{trans} with $b_{i,j}$ on the diagonal in the order the order of j varying most rapidly
$\mathbf{p}$	Column vector of length n_{trans} containing $p_{i,j}$ with j varying most rapidly
$\mathbf{p^{max}}$	Column vector of length n_{trans} containing $p_{i,j}^{max}$ (see $\mathbf{p}$)
$\boldsymbol{\tau}$	Column vector of length n_{trans} of Lagrange multipliers for the line equations
$\boldsymbol{\zeta}$	Column vector of length n_{trans} of Lagrange multipliers for the upper limits of the line power flows
$\boldsymbol{\xi}$	Column vector of length n_{trans} of Lagrange multipliers for the lower limits of the line power flows

Other symbols

c_{offer}	Total cost of supply based on offer prices
c_{trans}	Total cost of power transmission
$\mathbf{F}$	Matrix n_{node} by n_{offer} where the i^{th} row contains ones corresponding to the offers for generation in $\mathbf{g}$ at that node
$\mathbf{L}$	A lower triangular matrix
$\mathbf{R}$	Connection matrix n_{trans} by n_{node} for the network. The ℓ^{th} row corresponds to power line from node i to node j and the non-zero elements in that row are $r_{\ell,i} = -1$ & $r_{\ell,j} = 1$
$\mathbf{u}$	Vector of ones
$\mathbf{w}$	Vector of zeros except for one in last element
$\mathbf{X}'$	Matrix or vector $\mathbf{X}$ transposed
$\mathbf{X}^{@}$	Matrix or vector $\mathbf{X}$ with last row omitted
$\mathbf{X}^{\#}$	Matrix or vector $\mathbf{X}$ with last column omitted
$\mathbf{diag(x)}$	A diagonal matrix with the elements of the vector $\mathbf{x}$ on the diagonal
$[1_{i,j}]$	A matrix with one in the i,j position and zeros elsewhere

Table 8. Interconnected spring washers

LP cost = 16099.1
Generation payment = 16099.1
Demand payment = 205101.3

Node	Demand	$/MWh	Generation Max	Cost	Actual	w
1	400.5	125.00				
			1000.0	900	0.0	0.0
2	0.0	50.00				
			1000.0	50	2.9	0.0
3	0.0	20.00				
			1000.0	20	397.9	0.0
4	0.0	-55.00				
5	0.0	-242.50				
6	0.0	20.00				
			1000.0	20	399.8	0.0
7	0.0	125.00				
8	400.1	387.50				
			1000.0	900	0.0	0.0

Lines Start	End	Power transmission Max	Admit.	Cost	Actual	w
1	2	500.0	0.40	0	-200.8	0.0
2	3	500.0	1.00	0	-197.9	0.0
3	4	500.0	0.40	0	200.0	0.0
4	1	200.0	1.00	0	200.0	210.0
1	7	0.5	1.00	0	0.3	0.0
5	6	500.0	0.40	0	-200.0	0.0
6	7	500.0	1.00	0	199.8	0.0
7	8	500.0	0.40	0	200.1	0.0
8	5	200.0	1.00	0	-200.0	-735.0

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