Optimal tariff period determination†

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1 Background

CLP Power Hong Kong Ltd. has two different tariff periods, an off-peak period (21:00 to 09:00 and all day Sundays and Public Holidays) and a peak period (all other hours). The reason is that it costs CLP Power more to supply power when there is a greater demand for it (at peak time), so the higher tariff rate during the peak period of the day is meant to reflect that. A similar principle applies over the whole year. In Hong Kong, throughout summertime when it's hot and humid, demand is consistently higher than during other seasons due to extra air conditioning load. Obviously, in the extreme, perhaps the fairest way for CLP Power to charge their customers is, throughout the year, to have fixed seasonal, daily or hourly rates or even for example a continuous stream of rates published in advance say every half an hour, which fluctuate in proportion to changes in demand both daily and seasonally. Regarding the latter regime, for all except the most technically advanced customers, such pricing structures would be not be acceptable because customers would not be able to predict their monthly electricity bills with any certainty. Thus CLP Power sets its tariffs annually and tells its customers the tariff and peak and off peak periods before the year begins. The objective, therefore, is to set daily peak and off-peak periods accurately in advance so as to minimise the possibility, on the day, of customers being charged the peak rate when in actuality the load on the system is low and vice versa. This is not a profit maximisation problem. In a sense, it is a fairness optimization problem - setting tariffs that most accurately reflect the true real-time cost of supplying the power. We will assume overall, that customers will not change their total energy consumption habits if tariff periods are changed. In other words, we assume that the customers' response to changes in tariff rates is inelastic.

We separated the problem into two simpler problems. The first problem is to choose the seasonal tariff periods, and the second problem is to choose the daily tariff periods. During the study group, we mainly considered the first problem, which is simpler because there are just two seasonal tariff periods, peak and off-peak. In the second problem, we can have a maximum of four daily tariff periods.

2 Seasonal tariff period determination

Figure 1 shows an illustrative annual demand profile supplied by CLP Power Hong Kong Ltd. The year is made up of 365 data points, one for each day, but for many purposes it is convenient to treat the time of year as a continuous variable t in the range $0 \leq t \leq 1$.

The sample data in Figure 1 show substantial day-to-day fluctuations (shown in blue) around a smooth mean $f(t)$, shown in red. The value of f on day $365t$ is given by the average maximum demand on that day of all years prior, although computing this average is complicated by day-of-the-week variations, and also by festivities whose dates vary from year to year. In reality, the daily demand $Y(t)$, shown in blue, arises from sampling a continuous-in-time stochastic process at daily intervals.

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‡We would like to acknowledge the helpful comments from Dr. David Allwright at Oxford and from Ken Campbell at CLP Hong Kong Ltd.

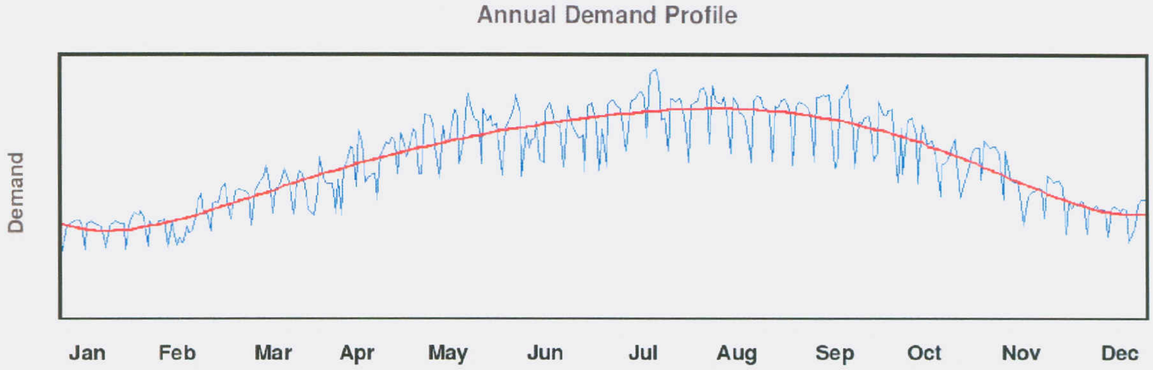

Figure 1: Illustrative annual demand profile supplied by CLP Power Hong Kong Ltd.

2.1 A simple model

We modelled the daily demand $Y(t)$ using uncorrelated day-to-day fluctuations around a prescribed seasonal average $f(t)$,

$$Y(t) = f(t) + X(t), \tag{2.1}$$

where for each day $356t$, $X(t)$ is an independently distributed normal random variable with zero mean and prescribed variance $\sigma^2(t)$,

$$X(t) \sim N\big(0, \sigma^2(t)\big). \tag{2.2}$$

The variance $\sigma^2(t)$ would also be calculated using data from previous years. It is a function of time because certain times of the year may be more volatile in terms of expected maximum daily demand. For example, the fact that the summer is more prone to extreme weather conditions, may make demand more volatile during that time of year. The actual demand on day $365t$ is thus distributed as

$$Y(t) \sim N\big(f(t), \sigma^2(t)\big). \tag{2.3}$$

To reiterate, this makes sense when $X(t)$ and $Y(t)$ are each 365 random variables labelled by a discrete index $t = n/365$ for $n = 1, 2, \ldots, 365$. Treating t as a continuous variable below enables a convenient approximation of sums over the 365 discrete points by integrals over a continous interval $0 \leq t \leq 1$.

Next, we assume that the peak period goes from time $t = a$ to time $t = b$, where $0 \leq a < b < 1$. The variables a and b are our control variables. The peak season in Hong Kong is the summer, so we should find that the interval $a \leq t \leq b$ corresponds to the middle half of the calendar year.

In order to determine the two tariff periods, we need to set target demand levels for the peak and off-peak periods, denoted by m_p and m_o, respectively. To minimize mismatch, i.e., lower demand (less than m_p) in the peak period or higher demand (more than m_o) in the off-peak periods, a natural choice is to set m_p as low as possible and m_o as high as possible. It is easy to see that without additional constraints, a possible solution which minimizes the mismatch is to set m_p equal to the lowest demand and declare the entire year as peak season, or vice verse, assign the highest demand to m_o and declare the entire year as off-peak period. Even though no mismatch occurs in either scenario, they are obviously not "sensible" solutions. Recall that the objective for setting the tariff periods is to reflect the fact that the cost of providing services is higher when the demand is also higher. To incorporate this additional "information", we assume that the levels m_p and m_o are set *a priori* at some "reasonable" values and our task at hand is to pick a and b so that the mismatch is minimized. In Appendix B, we describe another approach where m_p and m_o are determined as a part of the minimization problem, by minimizing the *averaged mismatch*, instead of the *total mismatch*.

Let $u(a,b)$ denote the total (sum of the areas of) mismatch expected in each peak or off-peak period,

$$u(a,b) = E\left[\int_0^a + \int_b^1 \left(Y(t) - m_o\right)^+ \mathrm{d}t\right] + E\left[\int_a^b \left(m_p - Y(t)\right)^+ \mathrm{d}t\right]. \tag{2.4}$$

The superscripts $+$ in (2.4) indicate the maximum of the bracketed quantity and zero. For example,

$$\left(Y(t) - m_o\right)^+ = \max\left(Y(t) - m_o, 0\right). \tag{2.5}$$

We thus want to minimise $u(a,b)$ over all possible values of a and b, finding

$$\min_{\{a,b\}} u(a,b). \tag{2.6}$$

The probability density function of our normally distributed random variable $Y(t)$ with mean $f(t)$ and variance $\sigma^2(t)$ is

$$\hat{p} = \frac{1}{\sigma(t)\sqrt{2\pi}} \exp\left(-\frac{(y - f(t))^2}{2\sigma^2(t)}\right). \tag{2.7}$$

Next, we want to find the expected values for (2.4). First, we have

$$E\left(Y(t) - m_o\right)^+ = \int_{m_o}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} (y - m_o) \exp\left(-\frac{(y - f(t))^2}{2\sigma(t)^2}\right) dy. \tag{2.8}$$

This integral may be evaluated as

$$\begin{aligned} E\left(Y(t) - m_o\right)^+ = \; & \frac{\sigma(t)}{\sqrt{2\pi}} \exp\left(-\frac{(m_o - f(t))^2}{2\sigma(t)^2}\right) \\ & - \frac{1}{2}\left(m_o - f(t)\right) \operatorname{erfc}\left(\frac{m_o - f(t)}{\sqrt{2}\sigma(t)}\right), \end{aligned} \tag{2.9}$$

in terms of the complementary error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt. \tag{2.10}$$

Similarly,

$$\begin{aligned} E\left(m_p - Y(t)\right)^+ &= \int_{-\infty}^{m_p} \frac{1}{\sigma\sqrt{2\pi}} (m_p - y) \exp\left(-\frac{(y - f(t))^2}{2\sigma(t)^2}\right) dy, \\ &= \frac{\sigma(t)}{\sqrt{2\pi}} \exp\left(-\frac{(m_p - f(t))^2}{2\sigma(t)^2}\right) \\ &\quad + \frac{1}{2}\left(m_p - f(t)\right) \operatorname{erfc}\left(-\frac{m_p - f(t)}{\sqrt{2}\sigma(t)}\right). \end{aligned} \tag{2.11}$$

We thus obtain

$$\begin{aligned} u(a,b) = & \int_0^a \left\{ \frac{\sigma(t)}{\sqrt{2\pi}} \exp\left(-\frac{(m_o - f(t))^2}{2\sigma(t)^2}\right) - \frac{1}{2}\left(m_o - f(t)\right) \operatorname{erfc}\left(\frac{m_o - f(t)}{\sqrt{2}\sigma(t)}\right) \right\} dt \\ & + \int_b^1 \left\{ \frac{\sigma(t)}{\sqrt{2\pi}} \exp\left(-\frac{(m_o - f(t))^2}{2\sigma(t)^2}\right) - \frac{1}{2}\left(m_o - f(t)\right) \operatorname{erfc}\left(\frac{m_o - f(t)}{\sqrt{2}\sigma(t)}\right) \right\} dt \\ & + \int_a^b \left\{ \frac{\sigma(t)}{\sqrt{2\pi}} \exp\left(-\frac{(m_p - f(t))^2}{2\sigma(t)^2}\right) + \frac{1}{2}\left(m_p - f(t)\right) \operatorname{erfc}\left(-\frac{m_p - f(t)}{\sqrt{2}\sigma(t)}\right) \right\} dt. \end{aligned} \tag{2.12}$$

In practice, these integrals over continuous time t could also be replaced by sums over 365 days using discrete daily values for $\sigma(t)$ and $f(t)$.

2.2 Numerical examples

In order to illustrate our approach, we now present several numerical examples. We choose the function $f(t)$ to be

$$f(t) = -\cos(2\pi t) + 4. \tag{2.13}$$

For $\sigma(t)$, we use three different values: σ =0, 1, and 2.

Example 1: $\sigma = 0$. In this case, there is no random fluctuation and we can solve the problem explicitly. It is easy to see that when $m_o = m_p = m$, a and b are the two roots of $f(t) = m$, in which case $u(a,b) = 0$, i.e., no mismatching occurs. For example, if $m = 4$, then $a = 1/4$ and $b = 3/4$, and if $m = 4.5$, we have $a = 1/3$ and $2/3$. On the other hand, if $m_p > m_o$, mismatch cannot be eliminated and in this case. Let $t_{o,1} < t_{o,2}$ be the two solutions of $f(t) = m_o$ and $t_{p,1} < t_{p,2}$ are the two solutions of $f(t) = m_p$. Since $t_{o,1} < t_{p,1}$ and $t_{p,2} < t_{o,2}$, it is easy to see that the solution exists and $t_{o,1} < a < t_{p,1}$ and $t_{p,2} < b < t_{o,2}$. For example, for $m_o = 3.5$ and $m_p = 4.5$, we have $t_{o,1} = 1/6$, $t_{p,1} = 1/3$, $t_{p,2} = 2/3$ and $t_{o,2} = 5/6$ and the solution is $a = 1/4$ and $b = 3/4$.

Having established the existence of a solution (and its dependence on the choice of m_o and m_p for the deterministic case), we can now examine the effect of the randomness on the choice of tariff periods.

Example 2: σ **=1, 2.** In this case the random fluctuation is normally distributed with a constant volatility 1 and 2. Again, the solution depends on the values of m_o and m_p. When $m_o = m_p = m$, we find that the solutions (a and b) are the same as that of the deterministic case. However, the minimized expected mismatch is no longer zero. For example, for $m_o = m_p = 4.5$, we have $a = 1/3$ and $b = 2/3$ and $u(a,b) = 0.173$. Changing the volatility from 1 to 2 does not change the value of a and b but $u(a,b)$ is now changed 0.511. When $m_o < m_p$, e.g., $m_o = 4$ and $m_p = 4.5$, we obtain $a = 0.290$ and $b = 0.710$ for both σ =1 and 2 while $u(a,b) = 0.265$ for $\sigma = 1$ and $u(a,b) = 0.636$ for $\sigma = 2$.

The insensitivity to random fluctuation of the solution (a and b) could be due to two possibilities: the symmetrical nature of the demand curve or constant volatility. We have, therefore, also tried a function for the variance that changes over time, namely, $\sigma = \exp(-25(t-0.5)^2) + 0.2$. However, our numerical tests show that the solution is almost identical to that of a constant volatility.

2.3 Implementation in practice

Since we do not have data from CLP Hong Kong Ltd., the numerical examples used in this report are for illustration purposes. However, the setup and methodology are applicable to CLP's problem. In order to use our approach, CLP needs to determine the function $f(t)$. This function represents the mean maximum daily load of all prior years on day $365t$. The resulting function will actually be a discrete function, but it will have to be interpolated to form a smooth function of t, $0 \leq t \leq 1$.[§] In our example, we used $f(t) = -\cos(2\pi t) + 4$, because did not have access to enough data with which we could calculate a better estimate for $f(t)$. However, this function does have the property that the mean maximum daily load is higher in the summer months, which is why we chose it for our examples. In a similar way, CLP must find the standard deviation function $\sigma(t)$, which represents the standard deviation of the maximum daily load of all prior years on day $365t$. Again, this function will need to be interpolated to form a smooth function, just like for $f(t)$.

[§]One could keep $f(t)$ as a discrete function, but then the integrals in the expected value calculation would have to be converted to sums, and our numerical procedure would need to be modified slightly.

3 A general setup for seasonal and daily tariff periods

In principle, we can generalise the approach in the previous section to determine both seasonal and daily tariff periods. We assume that the loading is given by

$$Y_t = f(t) + X_t \tag{3.1}$$

where X_t is a random variable that satisfies the following stochastic differential equation

$$dX_t = \alpha(X_t, t)dt + \beta(X_t, t)dB_t. \tag{3.2}$$

Here dB_t is a Brownian motion, i.e., $dB_t \sim N(0, dt)$. It follows that $p(x,t)$, the probability density function of X_t satisfies the Kolmogorov forward equation, cf. [1]

$$p_t + (\alpha p)_x - \frac{1}{2}(\beta^2 p)_{xx} = 0 \tag{3.3}$$

which can be solved with an appropriate initial condition. The corresponding probability is $P(x,t) = \int_{-\infty}^{x} p(z,t)dz$. Changing variable from X_t to $Y_t = X_t + f(t)$, we have $P(y,t) = \int_{\infty}^{y-f} p(z,t)dz = \int_{-\infty}^{y} p(z+f,t)dz$. Therefore, the probability density function for Y_t is given by $p(x+f,t)$.

3.1 Seasonal tariff period determination

In this case, the expectation (2.4) to be optimised becomes

$$\begin{aligned} u(a,b) =& \left[\int_0^a \int_{m_o - f(t)}^{\infty} (x + f(t) - m_o) p(x,t) \mathrm{d}x\mathrm{d}t\right] \\ &+ \left[\int_b^1 \int_{m_o - f(t)}^{\infty} (x + f(t) - m_o) p(x,t) \mathrm{d}x\mathrm{d}t\right] \\ &+ \left[\int_a^b \int_{-\infty}^{m_p - f(t)} \big(m - x - f(t)\big) p(x,t) \mathrm{d}x\mathrm{d}t\right]. \end{aligned} \tag{3.4}$$

3.2 Daily tariff period determination

If we only focus on a one day period, then we can apply the same approach for seasonal tariff period determination where there are two periods during the day. Over a longer period of time, one may still apply the same approach but allow for non-periodicity in the mean $f(t)$, i.e., the expectation to be minimised is $u(a,b) = \sum_{k=1}^{N} v(a,b,k)$ where

$$\begin{aligned} v(a,b,k) =& \left[\int_0^a \int_{m_o - f(t+k)}^{\infty} (x + f(t+k) - m_o) p(x,t+k) \mathrm{d}x\mathrm{d}t\right] \\ &+ \left[\int_b^1 \int_{m_o - f(t+k)}^{\infty} (x + f(t+k) - m_o) p(x,t+k) \mathrm{d}x\mathrm{d}t\right] \\ &+ \left[\int_a^b \int_{-\infty}^{m_p - f(t+k)} \big(m_p - x - f(t+k)\big) p(x,t+k) \mathrm{d}x\mathrm{d}t\right]. \end{aligned} \tag{3.5}$$

In this case, the unit of time is a day (instead of a year) and N is the number of days relevant to the optimisation process. If we assume that $f(t)$ is periodic on a yearly basis then $N = 365$. Note we have used the same notation for f and p when different time units are used, for simplicity.

4 Summary and recommendations

We considered the seasonal tariff period problem by optimising a functional $u(a, b)$ where a and b are controls which determine the peak and off-peak time intervals. The minimal u can be found numerically by combining inspection and using the first order (necessary) conditions. This could probably be done analytically, however, it involves solving a system of integral equations because the control variables appear in the limits of integration.

For the daily tariff period problem, we can apply the same approach for the seasonal tariff period problem when there are two tariff periods. For problems that are allowed more than two tariff periods, more work is needed.

To apply our approach, some followup studies are needed to be performed. First of all, data analysis has to be carried out to estimate the mean loading curve $f(t)$ as well as the volatility $\sigma(t)$, if the distribution of the random fluctuation turns out to be normal. We have also provided a framework for a more general stochastic demand process but the parameter values have to be estimated using existing data. Secondly, CLP also needs to decide which setup (minimize total or time averaged mismatch) is more suitable for their needs.

Using simple examples, we have shown in section 2 that by setting the target demand levels a priori and minimizing the total mismatch, we can find tariff periods which minimize the mismatch and the solution is quite insensitive to the random fluctuations in the daily demand. We have tried $\sigma = 0, 1, 2$ and $\sigma(t)$ as a given function of time, and the solution depends mainly on the choice of the target demand levels. In our second approach (minimizing the time averaged mismatch), which does not set the target demand levels a priori, the situation is different. When the target demand level for the peak tariff period is set to be the average of demands at the beginning and the end of the peak period, the volatility of the daily demand has a visible effect on the solution, cf. Appendix B. When $\sigma(t) = 1$, i.e., the volatility of the maximum daily load for any given day of the year is the same. It turned out that the answer is $a = 0.25$ and $b = 0.75$.When we considered a function $\sigma(t)$ which changed over the course of the year with higher volatility over the summer months, we obtained an optimal solution of $a = 0.150$ and $b = 0.850$. This essentially says that a higher standard deviation in the summer months means that we need to enlarge the peak tariff period.

References

[1] S.E. Shreve, *Stochastic Calculus for Finance II Continuous-Time Models*. Springer, NY, 2004.

A Probability density function for a mean-reverting Ornstein-Uhlenbeck process

We wish to relate the general model in Section 3 to the simple model constructed in Section 2.

For electricity load data provided by CLP, we postulate that the stochastic process follows

$$dX_s = -X_s ds + \eta(s) dB_s, \tag{A.1}$$

which is a special case of the general model with $\alpha = -X$ and $\beta = \eta$. The equation for the probability density function is

$$p_t - (xp)_x - \frac{\eta^2}{2} p_{xx} = 0. \tag{A.2}$$

We seek its solution in the form

$$p = \frac{1}{\sigma(t)\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2(t)}}, \tag{A.3}$$

and substituting it into (A.2) we obtain

$$\left(\frac{\sigma'}{\sigma} + 1 - \frac{\eta^2}{2\sigma}\right)\left(\frac{x^2}{\sigma^2} - 1\right) = 0, \tag{A.4}$$

which leads to

$$\frac{\sigma'}{\sigma} + 1 - \frac{\eta^2}{2\sigma} = 0, \tag{A.5}$$

or

$$\eta^2 = 2\sigma^2 + (\sigma^2)' \tag{A.6}$$

from which σ can be solved as

$$\sigma = \left(\sigma_0^2 e^{-2t} + \int_0^t e^{-2(t-s)} \eta^2 ds\right)^{\frac{1}{2}} \tag{A.7}$$

where $\sigma_0 = \sigma(0)$. In this case X_t has a normal distribution with zero mean and variance σ^2, corresponding to the simple case considered in Section 2. Note that if η is a constant, we obtain

$$\sigma = \left[\sigma_0^2 e^{-2t} + \frac{\eta^2}{2}\left(1 - e^{-2t}\right)\right]^{\frac{1}{2}} \tag{A.8}$$

which is stationary only when $\sigma_0 = \eta/\sqrt{2}$.

B An alternative setup for the seasonal tariff problem

We assume that periods of high demand in the two off-peak seasons $0 \le t \le a$ and $b \le t \le 1$ occur when $Y(t) \ge f(a)$ and $Y(t) \ge f(b)$ respectively. Since we may not have $f(a) = f(b)$, we further assume that a period of low demand in the peak season $a \le t \le b$ occurs when $Y(t) \le m$ with

$$m = \frac{f(a) + f(b)}{2}. \tag{B.1}$$

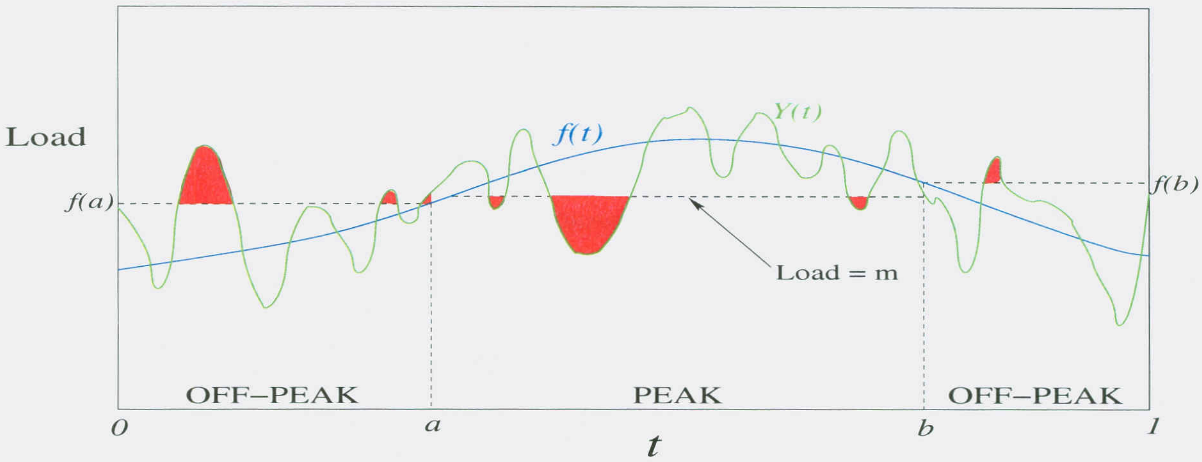

Figure 2: The division of the year into peak and off-peak seasons, illustrating the regions of unseasonably high or low demand that we wish to minimise.

However, this may or may not be the best possible choice. Figure 2 shows the peak and off-peak intervals, and the regions of unseasonably high or low demand. We wish to minimise the expected area in the regions shaded red. These regions are regions of *mismatch*, where maximum demand for that day is high but the off-peak tariff is charged or where the maximum demand for that day is low but the peak tariff is charged.

Let $u(a, b)$ denote the sum of the areas of mismatch expected in each peak or off-peak period, in proportion to the duration of the period,

$$\begin{aligned} u(a,b) = \quad & \frac{1}{a}E\Big[\int_0^a \big(Y(t) - f(a)\big)^+ \mathrm{d}t\Big] \\ & + \frac{1}{1-b}E\Big[\int_b^1 \big(Y(t) - f(b)\big)^+ \mathrm{d}t\Big] \\ & + \frac{1}{b-a}E\Big[\int_a^b \big(m - Y(t)\big)^+ \mathrm{d}t\Big]. \end{aligned} \tag{B.2}$$

The superscripts $+$ in (B.2) indicate the maximum of the bracketed quantity and zero. For example,

$$\big(Y(t) - f(b)\big)^+ = \max\left(Y(t) - f(b), 0\right). \tag{B.3}$$

The scaling by the length of the periods suppresses "trivial" solutions in which the mismatch during peak periods is minimised, for example, by having an off-peak period of zero duration.

We thus want to minimise $u(a, b)$ over all possible values of a and b, finding

$$\min_{\{a,b\}} u(a,b). \tag{B.4}$$

The probability density function of our normally distributed random variable Y with mean $f(t)$ and variance $\sigma^2(t)$ is

$$\hat{p} = \frac{1}{\sigma(t)\sqrt{2\pi}} \exp\left(-\frac{(y - f(t))^2}{2\sigma^2(t)}\right). \tag{B.5}$$

Next, we want to find the expected values for (2.4). First, we have

$$E\big(Y(t) - f(a)\big)^+ = \int_{f(a)}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}(y - f(a)) \exp\left(-\frac{(y - f(t))^2}{2\sigma(t)^2}\right) dy. \tag{B.6}$$

This integral may be evaluated as

$$E\big(Y(t)-f(a)\big)^+ = \frac{\sigma(t)}{\sqrt{2\pi}}\exp\left(-\frac{(f(a)-f(t))^2}{2\sigma(t)^2}\right) - \frac{1}{2}(f(a)-f(t))\,\mathrm{erfc}\left(\frac{f(a)-f(t)}{\sqrt{2}\sigma(t)}\right), \tag{B.7}$$

in terms of the complementary error function

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt. \tag{B.8}$$

Similarly,

$$\begin{aligned} E\big(m-Y(t)\big)^+ &= \int_{-\infty}^{m}\frac{1}{\sigma\sqrt{2\pi}}(m-y)\exp\left(-\frac{(y-f(t))^2}{2\sigma(t)^2}\right)dy, \\ &= \frac{\sigma(t)}{\sqrt{2\pi}}\exp\left(-\frac{(m-f(t))^2}{2\sigma(t)^2}\right) \\ &\quad + \frac{1}{2}(m-f(t))\,\mathrm{erfc}\left(-\frac{m-f(t)}{\sqrt{2}\sigma(t)}\right). \end{aligned} \tag{B.9}$$

We thus obtain

$$\begin{aligned} u(a,b) = {} & \frac{1}{a}\int_0^a\left\{\frac{\sigma(t)}{\sqrt{2\pi}}\exp\left(-\frac{(f(a)-f(t))^2}{2\sigma(t)^2}\right)-\frac{1}{2}(f(a)-f(t))\,\mathrm{erfc}\left(\frac{f(a)-f(t)}{\sqrt{2}\sigma(t)}\right)\right\}dt \\ & +\frac{1}{1-b}\int_b^1\left\{\frac{\sigma(t)}{\sqrt{2\pi}}\exp\left(-\frac{(f(b)-f(t))^2}{2\sigma(t)^2}\right)-\frac{1}{2}(f(b)-f(t))\,\mathrm{erfc}\left(\frac{f(b)-f(t)}{\sqrt{2}\sigma(t)}\right)\right\}dt \\ & +\frac{1}{b-a}\int_a^b\left\{\frac{\sigma(t)}{\sqrt{2\pi}}\exp\left(-\frac{(m-f(t))^2}{2\sigma(t)^2}\right)+\frac{1}{2}(m-f(t))\,\mathrm{erfc}\left(-\frac{m-f(t)}{\sqrt{2}\sigma(t)}\right)\right\}dt. \end{aligned} \tag{B.10}$$

In the real application these integrals over continuous time t could be replaced by sums over 365 days, using discrete daily values for $\sigma(t)$ and $f(t)$.

The solution can be obtained by evaluating the objective function $u(a,b)$ using Maple and applying the necessary conditions $\partial u/\partial a = \partial u/\partial b = 0$. Since the necessary conditions gives all extrema, inspection or second order derivative test is also needed to find the global minimum. Our numerical tests show that the solution of a and b is affected by the choice of volatility. For example, we have $a = 0.25$ and $b = 0.75$ for $\sigma = 1$. Again, this corresponds well with CLP's current seasonal tariff periods. In the second case, we set $\sigma = \exp(-25(t-0.5)^2)+0.2$, where $\sigma(t)$ is higher in the summer and lower in the winter, we find that the optimal solution requires an increase in the length of the peak season, given by $a = 0.150$ and $b = 0.850$. These numbers make sense because in this case, our choice of the function $\sigma(t)$ says that demand for power is more volatile in the summer, so it is more likely to get large outliers. Consequently, the peak interval needs to be widened to capture as many of these outliers as possible. Attached below is a Maple worksheet for evaluating and plotting $u(a,b)$.

Maple worksheet.

```
>restart;

>assume(sigma(t)>0,t>0);

>innerint1:=int((y-f(a))*exp(-(y-f(t))^2/(2*sigma(t)^2)),y=f(a)..infinity);

>innerint2:=int((y-f(b))*exp(-(y-f(t))^2/(2*sigma(t)^2)),y=f(b)..infinity);
```

```
>fun1:=(a,b)->1/(sqrt(2*Pi))*int(1/sigma(t)*innerint1,t=0..a)/a;

>fun2:=(a,b)->1/(sqrt(2*Pi))*int(1/sigma(t)*int(((f(a)+f(b))/2-y)*exp(-(y-f(t))^2/(2*sigma(t)^2)),y=-infinity..(f(a)+f(b))/2),t=a..b)/(b-a);

>fun3:=(a,b)->1/(sqrt(2*Pi))*int(1/sigma(t)*innerint2,t=b..1)/(1-b);

>sigma:=t->exp(-(5*t-5/2)^2)+0.2;

>f:=t->-cos(2*Pi*t)+4;

>with(plots):

>p1:=plot3d(evalf(fun1(a,b)+fun2(a,b)+fun3(a,b)),a=0..1,b=a..1,axes=boxed, axesfont=[HELVETICA,12],labelfont=[TIMES,ITALIC,16],labels=[a,b,u]):

>display(p1);

>best:=100;

>for a from 0.005 to 1 by 0.005 do

for b from a to 1 by 0.005 do

candidate:=evalf(fun1(a,b)+fun2(a,b)+fun3(a,b)):

if candidate<best then

abest:=a:

bbest:=b:

best:=candidate:

end if:

end do:

end do:

print(abest);

print(bbest);
```