Optimization of Collateral Value Distribution

Ivica Bošnjak* Nataša Krejić* Marjan Milanović†
Mladen Nikolić‡ Marko Petković† Dragan Rakić†

99th European Study Group with Industry, Novi Sad, February 2014

1 Problem statement

The problem was presented by OTP Bank Serbia. Loan Loss Provisioning (LLP) is an amount of reserve that banks "put aside" to cover loss in case that loan goes in default, meaning that clients do not repay it. It is a safety buffer for preserving banks liquidity and capital adequacy. On the other hand, the Loan Loss Provisioning is a cost. In the Profit and Lost statement of banks, LLP decreases profit. It is a good tool/mechanism for risk management, but also expensive one, and that is why it is important for banks to optimize it in every possible way.

LLP is calculated based on:

- Loan exposure/amount (€)
- Loan probability of default (%)
- Loan collaterals value (€)

and is defined by the following relation:

LLP = Unsecured part of the loan × Probability of default for the loan

Unsecured part of the loan represents Exposure/amount (€) of the loan decreased for the Value (€) of the loan collaterals. The definition of LLP can therefore be rewritten as follows:

LLP = (Loan Exposure - Loan collaterals value) × Prob. of default for the loan

where

- LLP is subject of minimization

*University of Novi Sad, Faculty of Sciences, email:{ivb@dmi.uns.ac.rs, natasa@dmi.uns.ac.rs}
†University of Niš, Faculty of Natural Sciences and Mathematics, email:{makimozak@gmail.com, dexterofnis@gmail.com, rakic.dragan@gmail.com}
‡University of Belgrade, Faculty of Mathematics, email:nikolic@matf.bg.ac.rs

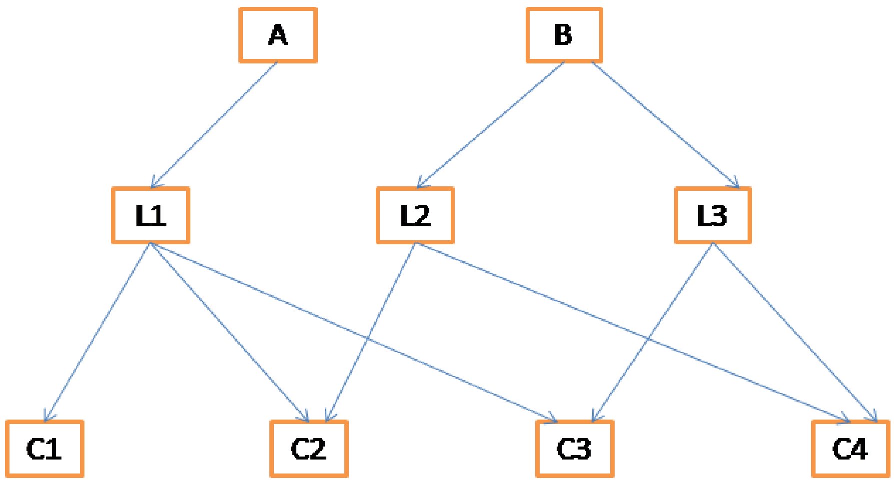

Figure 1: Possible connections between loans and collaterals

- Loan collateral value is subject of distribution optimization
- Loan exposure/amount is a given value for each loan
- Probability of default is a given value for each loan

The aim of optimization is to distribute collateral value to the connected loans, in a way to minimize amount of LLP. It can be done easily on a one loan level, but creating a universal algorithm that is applicable to all loans and all collaterals on the Bank portfolio level, is the goal to be achieved.

In the computation of LLP, the appraised market value of collateral is not used directly. It is decreased for the amount of previous encumbrances (if there is any) and after that, decreased with the corrective factor in order to calculate the bank Accepted value of a collateral. The Accepted value of a collateral is the value that should be distributed to the connected loans in an optimal way.

Accepted value = (appraised value - previous encumbrances) $\times$ **corrective factor**

where:

- Appraised value is the market value from the appraisal of licensed appraiser, validated by the Bank experts
- Previous encumbrances are the amount of higher rank mortgage/pledge which are inscribed in favor of third party
- Corrective factor is the factor (in %) defined by the Bank, indicating which part of appraised value the Bank accept as reasonable in case of potential collection from it by selling it.

2 Mathematical model

In order to construct a mathematical model, we need to define all given values and their mutual correlation.

- Loan exposures (amounts) will be denoted by $L_1, L_2, \ldots, L_n$
- Appraised market values of collaterals are $C_1, C_2, \ldots, C_m$
- For every loan, we can consider the set of all collaterals that can be used to secure it: $A(j) = \{k : C_k \text{ serves as a collateral for } L_j\}$
- Probabilities of default $p_j, j = 1, \ldots, n$. These probabilities depend on the ranking of client, but we may assign a corresponding probability to any loan. So, probability p_j corresponds to loan L_j.
- Correction coefficients $w_{ij}, i = 1, \ldots, m, j = 1, \ldots, n$. Correction coefficients depend not only on collaterals, but on loans as well. So, w_{ij} is the corrective factor applied to the part of collateral C_i, which is used for loan L_j. It is convenient and natural to set $w_{ij} = 0$ if $i \notin A(j)$.
- Useful value of a collateral is defined by

  $$\tilde{C}_i = C_i - \text{previous encumbrances.}$$

  The correction coefficients are applied only to Useful value of a collateral.

We need to distribute collateral values to the connected loans. By x_{ij} we will denote the percentage of $\tilde{C}_i$ used for covering L_j. Therefore, x_{ij} is a number between 0 and 1. There is a natural constraint for these numbers:

$$x_{ij} = 0 \text{ if } w_{ij} = 0.$$

Unsecured value of the loan L_j can now be calculated as:

$$L_j - \sum_{i=1}^{m} x_{ij} w_{ij} \tilde{C}_i.$$

The problem of minimizing the Loan Loss Provision can be formulated as the following linear programming problem:

$$\min_{\substack{x_{ij}: i \in A(j) \\ j \in \{1,\ldots,n\}}} \quad f = \sum_{j=1}^{n} \left(L_j - \sum_{i=1}^{m} x_{ij} w_{ij} \tilde{C}_i \right) p_j \tag{1}$$

$$\text{s.t.} \quad \sum_{i=1}^{m} x_{ij} \leq 1, \; i = 1, \ldots, m \tag{2}$$

$$L_j - \sum_{i=1}^{m} x_{ij} w_{ij} \tilde{C}_i \geq 0, \; j = 1, \ldots, n \tag{3}$$

$$x_{ij} \geq 0, \; i \in A(j), j = 1, \ldots, n \tag{4}$$

The problem above is easily solvable by standard LP methods. Given that the dimensions should not be too large, a simple application of the simplex algorithm will provide a solution to the above problem.

The solution to this problem may not be unique. For instance, whenever it is possible to distribute collateral values so that LLP is equal 0, it is reasonable to expect multiple optimal solutions. If this is the case, another problem arises: how to determine the "best" optimal solution? In other words, we need to minimize the overall amount of distributed collateral value. This problem can be expressed and solved as another LP problem:

$$\min_{\substack{x_{ij}:i\in A(j)\\ j\in\{1,\ldots,n\}}} \quad g=\sum_{j=1}^{n}\sum_{i=1}^{m}x_{ij}\tilde{C}_i \tag{5}$$

$$\text{s.t.} \quad \textstyle\sum_{j=1}^{n}\left(L_j-\sum_{i=1}^{m}x_{ij}w_{ij}\tilde{C}_i\right)p_j=\text{minimum}f$$

$$\sum_{i=1}^{m}x_{ij}\le 1,\ i=1,\ldots,m$$

$$L_j-\sum_{i=1}^{m}x_{ij}w_{ij}\tilde{C}_i\ge 0,\ j=1,\ldots,n$$

$$x_{ij}\ge 0,\ i\in A(j), j=1,\ldots,n$$

Here, minimum f is the optimal value of LLP obtained by solving the previous linear programming problem. The other constraints remain the same.

3 Examples

The bank also submitted eight examples from practice, with exact values, to be solved. Here we present the solutions of problems 3, 7 and 8. The problem were solved using the simplex method implemented as a built-in function in Mathematica. More details on the simplex method can be found in the references listed at the end of this report.

- Example 3

There are two clients, A and B, A has the internal rating 2 and B has the internal rating 3. A has one loan $L_1 = 350$, while B has $L_2 = 120, L_3 = 95, L_4 = 4$. Again, all amounts are in thousands. Available collaterals are $C_1 = 850, C_2 = 320, C_3 = 250$. The collaterals are of different types, there are no previous encumbencies but the bank has different order of mortgagees/pledges on these collaterals. The relationship between loans and collaterals are such that $A_1 = \{1,2\}, A_2 = \{1,2\}, A_3 = \{1,2\}, A_4 = \{2,3\}$ but the order of mortgages is different within these sets. Thus the matrix of correction coefficients is the following

$$\begin{bmatrix} 0.5 & 0.6 & 0.5 & 0 \\ 0.5 & 0.4 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}.$$

The set of default probabilities we used in this example is $p_1 = 0.1151, p_2 = 0.2235, p_3 = 0.2235, p_4 = 0.2235$. The solution x_{ij} is given in the matrix below,

$$\begin{bmatrix} 13/17 & 4/17 & 0 & 0 \\ 25/160 & 0 & 95/160 & 4/160 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this example the total of all loans is completely covered by such distribution of collaterals C_1 and a part of C_2, while C_3 is not used at all. Applying the correction coefficients we can easily see that $L_1 = 650 \cdot 0.5 + 50 \cdot 0.5 = 350,\ L_2 = 200 \cdot 0.6 = 120,\ L_3 = 190 \cdot 0.5 = 95$ and $L_4 = 8 \cdot 0.5 = 4$. However the solution is not unique and other combinations are possible but with the same results.

- Example 7.

In this example there is a single client with internal rating 1 and 6 loans $L_1 = 40, L_2 = 50, L_3 = 40, L_4 = 87.5, L_5 = 100$ and $L_6 = 30$. All amounts are in thousands. The available collaterals are of different types with the following values $C_1 = 5.5, C_2 = 150, C_3 = 14$ and $C_4 = 250$. There are no previous encumbrances thus the useful values of collaterals are the same, $\tilde{C}_1 = 5.5, \tilde{C}_2 = 150, \tilde{C}_3 = 14, \tilde{C}_4 = 250$. The collaterals are of different types implying that the haircut coefficients will be different. All collaterals can be used to cover any of the loans so $A_1 = \ldots A_6 = \{1, 2, 3, 4\}$. We assumed here that $p_j = 0.1, j = 1, \ldots, 6$. The matrix of correction coefficients is given by

$$\begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.6 & 0.6 & 0.6 & 0.5 \\ 0.4 & 0.4 & 0.5 & 0.5 & 0.5 & 0.4 \\ 0.5 & 0.5 & 0.6 & 0.6 & 0.6 & 0.5 \end{bmatrix}.$$

Solving this problem with the simplex method we get the following solution matrix with x_{ij} elements

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 11/30 & 19/30 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.13 & 0 & 0 & 0.20332 & 0.66668 & 0 \end{bmatrix}.$$

In other words $\tilde{C}_1 = 5.5$ will be used to cover L_1, $\tilde{C}_2 = 150 = 55+95$ will be used to cover L_3 (with 55) and L_4 (with 95). Whole $\tilde{C}_4 = 14$ will be used to cover L_3 and $\tilde{C}_4 = 250 = 32.5 + 50.83 + 166.67$ will be used to cover L_1 (with 32.5), then L_4 (with the amount of 50.83) and L_5 (with 166.67). Applying the corresponding correction coefficients (which depend on the type of collateral and the order of bank's mortgage on each collateral) we get that the uncovered part of the loans consists of uncovered part of L_1 which is equal to $21 = 40 - 5.5 \cdot 0.5 - 32.5 \cdot 0.5$ and $L_2 = 50, L_6 = 30$ which are completely uncovered. In total the unsecured part of all loans is 101. Changing the probabilities of default one would get a different result.

- Example 8.

In this example we have three clients, A with the internal rating 3, B with the internal rating 4 and C with the internal rating 5. The loans of A are $L_1 = 100, L_2 = 300, L_3 = 150, L_4 = 200$. B has the following loans $L_5 = 10, L_6 = 20, L_7 = 100, L_8 = 75$, while C has the loans $L_9 = 50, L_{10} = 40$. Again, all amounts are in thousands. The available collaterals are $C_1 = 150, C_2 = 280, C_3 = 260, C_4 = 100, C_5 = 35, C_6 = 47, C_7 = 455, C_8 = 183, C_9 = 372, C_{10} = 4, C_{11} = 80, C_{12} = 150, C_{13} = 300$. There are previous encumbencies on C_5 and C_{13}, so $\tilde{C}_5 = 7$ and $\tilde{C}_{13} = 47$, while for all other collaterals we have $\tilde{C}_j = C_j$. The set A that define loan-collateral relationships are the following: $A_1 = \{1, 2, 3, 4, 6, 7, 10\}, A_2 = \{1, 2, 4, 6, 8, 9\}, A_3 = \{2, 3, 4, 5, 7, 9\}, A_4 = \{1, 2, 4, 5, 8\}, A_5 = \{7, 10, 12\}, A_6 = \{7, 12\}, A_7 = \{7, 11, 12\}, A_8 = \{7, 11, 12\}, A_9 = \{8, 12, 13\}, A_{10} = \{8, 10, 12, 13\}$. The order of mortgages is again different for different loans and the collaterals are of different types so the matrix of correction coefficients is given by

$$\begin{bmatrix}
0.5 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.6 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.6 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.6 & 0.6 & 0.6 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.4 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.4 & 0 & 0.4 & 0 & 0.4 & 0.4 & 0.5 & 0.4 & 0 & 0 \\
0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0.6 & 0.6 \\
0 & 0.7 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.6 & 0.6 & 0.7 & 0.6 & 0.7 & 0.7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0.6
\end{bmatrix}.$$

We assumed that the default probabilities for the loans $L_1, \ldots, L_{10}$ are the following, $p_1 = p_2 = p_3 = p_4 = 0.1, p_5 = p_6 = p_7 = p_8 = 0.2$ and $p_9 = p_{10} = 0.3$. Applying the simplex algorithm we get the following solution matrix.

$$\begin{bmatrix}
0 & 7.2/150 & 0 & 70.8/150 & 0 & 0 & 0 & 0 & 0 & 0 \\
156.67/280 & 0 & 0 & 6.2/280 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/26 & 0 & 25/26 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
00 & 00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.4 & 0 & 0.4 & 0 & 0.4 & 0.4 & 0.5 & 0.4 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0.6 & 0.6 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 13.33/150 & 33.33/150 & 45/150 & 58.33/150 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 83.33/253 & 66.67/253
\end{bmatrix}.$$

Again, all loans are covered and some collateral are not used at all. The solution in this case is not unique so the second LP (5) is used to get this particular solution.

References

[1] Predrag Stanimirović, Nebojša Stojković, Marko Petković, Matematičko programiranje, University of Niš, Faculty of Science and Mathematics, 2007

[2] Robert Vanderbei, Linear Programming: Foundations and Extensions (4th ed.), Springer, 2014.